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☞All Business Communications should be addressed to the Editor and Manager, University, La., or P. O. Box 1322, Baton Rouge, La.

This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematics.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Engineering Mathematics

It is a well recognized fact that engineers need at least one course in the calculus. In fact, what would physical chemistry, mechanics, alternating current theory and hydro-dynamics, to mention only a few subjects, be without calculus? Does the engineer need more than this minimum first course? In view of the rapid advance of technical applications of mathematics it seems wise to encourage engineering students to cover as much ground as time permits. Maurice Frechet in his *Theorie Elementaire des Equations Differentielles,* Tournier et Constans, Paris, 1937, predicts that the more or less simple differential equations met in present applications will be replaced by more and more complicated ones as the technique of measurement improves: the present imperfection of measurement has permitted only a rough approximation of the known functions appearing in the equations, often by first degree polynomials.

A few examples will serve to indicate how much one field (electrical engineering) has changed in the course of less than thirty years. The representation of steady state alternating current phenomena by the use of complex numbers plays a large role in the work of junior and senior electrical engineering undergraduates. Many instructors did not have that training as undergraduates. Rough approximations for long transmission lines were given not many years ago. *Now* it is necessary to use frequently the exact solution, since technical improvements and high voltage lines require much greater accuracy. Operational calculus ("D" operators and their generalizations) is now widely used (line integrals in the complex plane, Laplace transforms, Fourier transforms). Many engineering schools are giving courses in tensor analysis, something unheard of a few years ago. Empirical design of large circuit breakers has given way to mathematical treatment—the older design would have produced circuit breakers too large to go through railroad tunnels!

A short list follows of a few books and articles treating engineering mathematics. Some (not all) of the topics covered are mentioned specifically:

W. F. Durand, *Aerodynamic Theory*, Vol. I (1934), Vol. II (1935), Springer, Berlin.

Functions of a complex variable, conformal representation, potential, Fourier series, partial differential equations, vector analysis.

H. Villat, *Lecons sur l'Hydrodynamique*, Gauthier-Villars, Paris, 1929. (Mechanics of Fluids, University of Paris).

Functions of complex variables, potential, conformal representation, elliptic functions, Legendre functions.

R. E. Doherty and E. G. Keller, *Mathematics of Modern Engineering*, Vol. I, John Wiley and Sons, New York, 1936.

Ordinary linear differential equations, Fourier series, Græffe's root squaring method, numerical integration, vector analysis, partial differential equations, Heaviside's operational calculus, functions of a complex variable.

G. Kron, *General Electric Review*, a series of articles on *The Application of Tensors to the Analysis of Rotating Electrical Machinery, from 1935 to 1938.*

Tensors, dyadics, matrices, spinor analysis, analysis situs. The linear form of the equations involved leads naturally to these generalizations of the complex numbers used to represent steady state alternating current phenomena. Analysis situs enters in the discussion of electrical networks. The transformations involved correspond to connections of the elements of a circuit.

N. W. McLachlan, *Bessel Functions for Engineers*, Clarendon Press, Oxford, 1934.

Bessel, hypergeometric, gamma, Struve functions with applications to acoustics, electrical engineering (circuit theory, calculation of inductance, artificial lines, electric furnaces, skin effect, eddy currents). Four pages of references to engineering applications of Bessel functions are given.

M. Walker, *Conjugate Functions for Engineers*, Oxford University Press, London, 1933.

Applications are made in the form of thirteen examples with a discussion of such problems as distribution of flux in air gaps, the fringing flux between two poles of a generator, armature slots, pole pieces with rounded tips.

R. Rothe and others, *Funktionentheorie und ihre Anwendung in der Technik*, Springer, Berlin, 1931.

A brief resume of the theory of functions of a complex variable followed by five articles written by engineers on various applications to technical problems.

J. R. Carson, *Electric Circuit Theory and the Operational Calculus*, McGraw-Hill, New York, 1936.

Heaviside operational calculus, Stieltjes integrals, integral equations, Laplace transforms, divergent series.

The Heaviside operational calculus is widely used by electrical engineers. A discussion of some of the many ramifications of the subject may be found in

H. T. Davis, *The Theory of Linear Operators*, Principia Press, Bloomington, 1936.

H. Jeffreys, *Operational Methods in Mathematical Physics*, Cambridge University Press, London, 1931.

A wide range of engineering problems is treated in H. Bateman, *Partial Differential Equations of Mathematical Physics*, Cambridge University Press, London, 1932.

The field of engineering mathematics seems somewhat neglected by American mathematicians, who are more interested in the abstract. Dean Birkhoff pointed out in his lecture *Fifty Years of American Mathematics*, Semi-centennial Addresses of the American Mathematical Society, New York, 1938, that "Aside from the Journal of Mathematics and Physics there is as yet no journal directed toward applied mathematics." There is in print a mass of information which needs to be edited, often entirely rewritten with due regard to present mathematical knowledge. Who is better fitted for this tremendous labor than the engineer well versed in mathematics?

WILLIAM E. BYRNE.
Virginia Military Institute.

Line Motion and Trisection

By Robert C. Yates
University of Maryland

The Archimedian principle of *verging** to the solution of the Trisection Problem is one familiar to all beginners in elementary geometry whose curiosity has led them into the dark woods of this famous problem. With center at the vertex of the given arbitrary angle AOB describe the unit circle, meeting the sides of the angle at A and B. It was clear to Archimedes that if a line AC could be drawn meeting OB

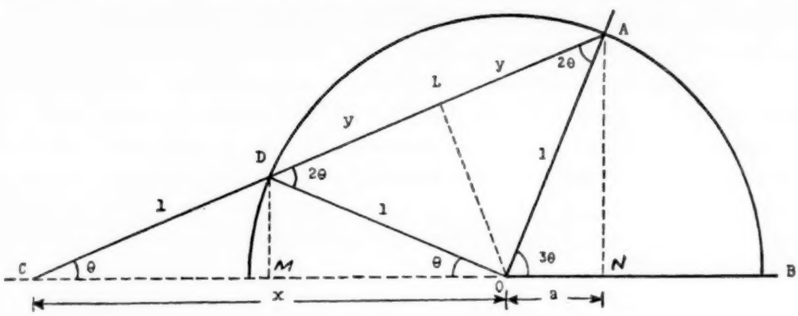

Fig. 1.

(produced) at C such that $DC=1$, then the problem was solved. For, if $\sphericalangle AOB=3\theta$, it follows directly from an inspection of Fig. 1 that $\sphericalangle ACO=\theta$. But drawing the line AC meant first locating the point C or, which is the same thing, finding the distance $OC=x=2\cos\theta$. Let $2y$ denote the distance AD and a the projection of OA upon the side OB. From similar triangles CMD, CNA, and CLO, all right triangles with equal angles at C, we find:

$$x/2=(x+a)/(1+2y)=(1+y)/x,$$

which give on eliminating y:

$$\boxed{x^3-3x-2a=0,}$$

the Trisection Equation. Thus the problem Archimedes hoped to solve by straight-edge and compass was of the third degree, generally

*Greek: *neusis*, German: *Einschiebung*.

irreducible and impossible of solution in every instance by these means.

Surprisingly little more in the way of equipment is needed, however, for the solution of problems of higher degree than the second. It has been shown, for instance, that the compass and *graduated* ruler are sufficient to solve all third and fourth degree problems.* Two right triangles, of the sort that are found in every draughting office, will trisect an arbitrary angle and give simple constructions for regular polygons. Such things are admirably discussed in Adler's *Geometrischen Konstruktionen*, Hilbert's *Foundations of Geometry*, etc. Professor Bussey has recently urged that these non-classical tools be discussed more frequently in courses on plane geometry in an effort to stem the ever-flowing tide of "circle-squarers" and "trisectors."†

Returning to Fig. 1, it is apparent that if OD and AC are links, all that is required for a trisection machine is for the point C to be restricted to motion along the line OB. This demand can be met in a variety of ways,‡ all utilizing the idea of link motion. The simplest arrangement is adapted from an elegant apparatus composed of only five bars due to Hart.§ This is shown in Fig. 2. Let O and O' be fixed points and select rods with the following lengths:

$$PA = OA = 1, \quad PB = b, \quad BO' = c.$$

From the figure:

$$\cos \sphericalangle PAO = \frac{2-(PO)^2}{2}, \quad \cos \sphericalangle PBO' = \frac{b^2+c^2-(PO')^2}{2bc}.$$

If, in some way, these two angles can be made equal throughout all deformations of the linkage, then from the preceding equations:

$$2-(PO)^2 = \frac{b^2+c^2-(PO')^2}{bc}.$$

Furthermore, if $bc=1$,‡ then the difference of the squares of the distances of the moving point P from the two fixed points, O and O', is constant:

$$(PO') - (PO)^2 = b^2+c^2-2 = (b-c)^2 = (1-b^2)^2/b^2.$$

*That is, problems leading to cubic and quartic equations whose coefficients represent line segments.

†*American Mathematical Monthly*, Vol. 43 (1936), pp. 265-280.

‡See Kempe, A. B.: *On a General Method of producing exact Rectilinear Motion by Linkwork*, Proc. Lon. Roy. Soc., XXIII (1875), pp. 565-577. Also the writer's paper: *An Ellipsograph*, National Mathematics Magazine, XII, (1938), pp. 213-215.

§Hart, H.: *On Some Cases of Parallel Motion*, Proc. Lon. Math. Soc., VIII (1877), p. 288.

‡In what follows we take $b<1$.

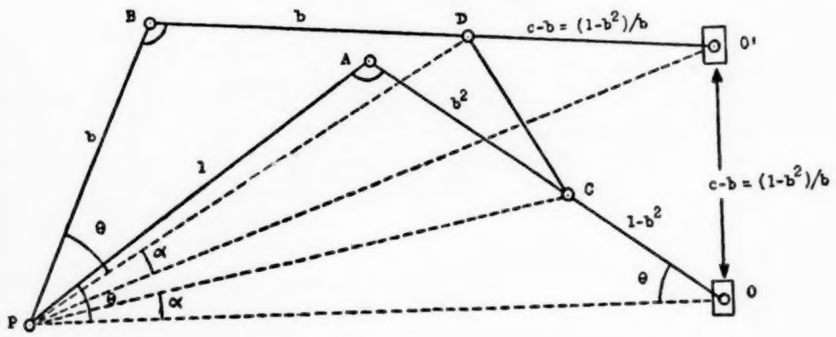

Fig. 2.

Accordingly, P will describe a line perpendicular to OO'. If the fixed distance OO' be taken equal to $(c-b)=(1-b^2)/b$, the line will pass through O and, incidentally, any point of AP will trace an ellipse.

Now the mechanical arrangement necessary to insure that angles PAO and PBO' be always equal would seem indeed difficult to realize. However, surprisingly enough, such is not the case, for there are two points, C on OA and D on $O'B$, *which remain at a constant distance apart.* Thus the addition of the single bar CD is sufficient. Let the points C and D be selected so that

$$AC=b/c=b^2; \qquad BD=b.$$

We have, from the similar isosceles triangles PBD and PAO (base angles Θ):

$$PD=2b\cos\Theta, \qquad\qquad PO=2\cos\Theta.$$

Further, from PBO':

$$(PO')^2=b^2+c^2+2bc\cos 2\Theta=b^2+1/b^2+2\cos 2\Theta,$$

and from PAC:

$$(PC)^2=1+b^4+2b^2\cos 2\Theta.$$

Thus

$$(1)\qquad PD=b\cdot(PO),\quad PC=b\cdot(PO'),\quad OC=b\cdot(DO').$$

The sides of triangle PDO' are then proportional to those of CPO with equal angles, α, at P. Accordingly,

$$\sphericalangle DPC=\sphericalangle O'PO,$$

since each is $(\alpha+\sphericalangle O'PC)$. It follows from (1) that triangles PDC

and *POO′* are similar and *PD* is therefore always perpendicular to *DC*. We have then

$$DC = (1 - b^2),$$

a constant.

It is evident that by producing *PA* to *M* so that *AM* is at least two units in length we have the mechanical trisector shown in Fig. 3.

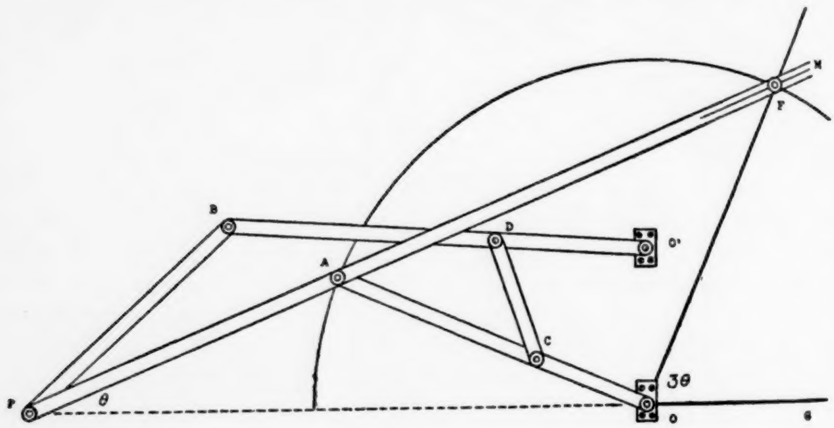

Fig. 3.

In application it is necessary to place one side of the given angle along *OG*, which is perpendicular to *OO′*, then bring the bar *PM* coincident with *F*, the point of intersection of the unit circle with the other side of the angle.*

*A bar *OF* may be attached with *F* as a joint sliding in the bar *PF*. It would then, of course, be unnecessary to draw the unit circle.

Solution of Numerical Equations by Use of the Circle of Curvature

By Richard J. Wells
Ohio State University

As the following diagram indicates, this method is very similar to Newton's method, the tangent being replaced by the circle of curvature. The procedure may be applied to all numerical algebraic or transcendental equations which satisfy the following conditions:

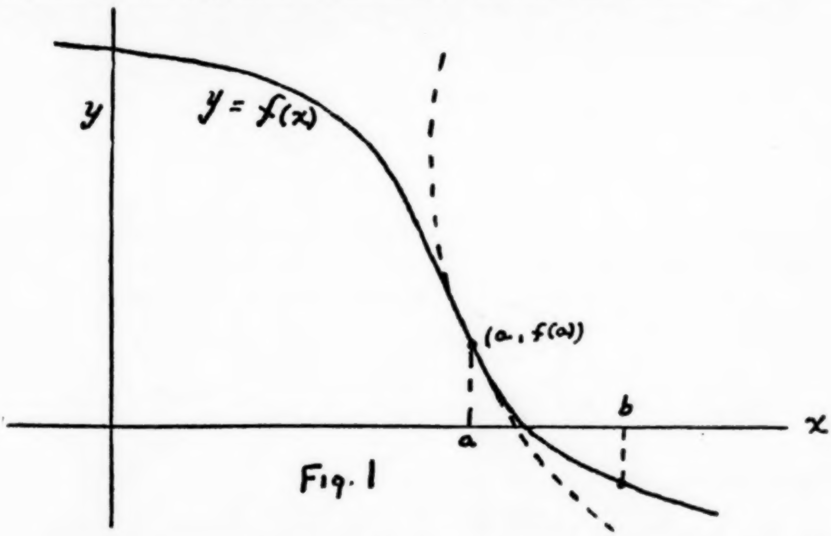

Fig. 1

1. The equation $f(x)=0$ has a single real irrational root within an interval $(a \leqq x \leqq b)$.
2. The function $f(x)$ is single-valued and continuous in this interval.
3. The derivatives $f'(x)$ and $f''(x)$ are single-valued and continuous, and have no real zero in this interval.

From elementary calculus, the equation of the circle of curvature at the point $(a,f(a))$ is:

$$\left[x-a+f'(a)\frac{1+[f'(a)]^2}{f''(a)}\right]^2 + \left[y-f(a)-\frac{1+[f'(a)]^2}{f''(a)}\right]^2 = \frac{[1+[f'(a)]^2]^3}{[f''(a)]^2}. \tag{1}$$

Setting $y=0$, and solving for the x-intercept nearest to $x=a$, we have:

$$(2)\qquad x=a-q\left\{t+\frac{1}{2!}t^2+\frac{1\cdot 3}{3!}t^3+\frac{1\cdot 3\cdot 5}{4!}t^4+\cdots\cdots\right.$$
$$\left.+\frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}t^n+\cdots\cdots\right\}$$

where,
$$q=f'(a)\,\frac{1+[f'(a)]^2}{f''(a)}$$

$$t=\frac{f(a)f''(a)}{[f'(a)]^2\,[1+[f'(a)]^2]}\left\{\frac{f(a)f''(a)}{2[1+[f'(a)]^2]}+1\right\}$$

It is easily shown that the series (2) converges if,
$$|t|<\frac{1}{2}$$

Since, by hypothesis, $f'(a)\neq 0$, this condition can always be fulfilled by proper choice of a.

From (2), it is apparent that the solution can be expressed as:

$$(3)\qquad x=a+t_1+t_2+t_3+\cdots\cdots+t_n+\cdots\cdots$$

Letting, $\quad f(a)=a_1;\quad \dfrac{f'(a)}{1!}=a_2;\quad \dfrac{f''(a)}{2!}=a_3$

we have:

$$(4)\qquad t_1=-\frac{a_1a_3+a_2^2+1}{a_2^2+1}\cdot\frac{a_1}{a_2};$$

$$t_2=-\frac{(a_1a_3+a_2^2+1)^2}{(a_2^2+1)^3}\cdot\frac{a_1^2}{a_2^3}\cdot a_3.$$

$$t_3=-\frac{(a_1a_3+a_2^2+1)^3}{(a_2^2+1)^5}\cdot\frac{a_1^3}{a_2^5}\cdot 2a_3^2;$$

$$t_4=-\frac{(a_1a_3+a_2^2+1)^4}{(a_2^2+1)^7}\cdot\frac{a_1^2}{a_2^7}\cdot 5a_3^3;$$

$$t_n=-\frac{1\cdot 3\cdot 5\cdot 7\cdots\cdots(2n-3)}{n!}\cdot\frac{(a_1a_3+a_2^2+1)^n}{(a_2^2+1)^{2n-1}}\cdot\frac{a_1^n}{a_2^{2n-1}}(2a_3)^{n-1}.$$

The above substitutions are made so that the quantities, a_1, a_2, and a_3 may be obtained by synthetic division when an algebraic equation of the form:

$$p_0x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+p_4x^{n-4}+\cdots+p_n=0$$

is being solved.

If t is small, the series (2) converges very rapidly, and in most cases a sufficiently accurate result may be obtained by neglecting all terms which are of a second or higher degree in t. Then from (3) and (4), the result is:

$$x=a-\frac{a_1a_3+a_2^2+1}{a_2^2+1}\cdot\frac{a_1}{a_2} \tag{5}$$

So far, we have been concerned only with finding an approximation to the $x-$intercept of the circle of curvature. In order to see how closely this value approximates the actual root of the equation, let us consider the following approximate geometrical derivation:

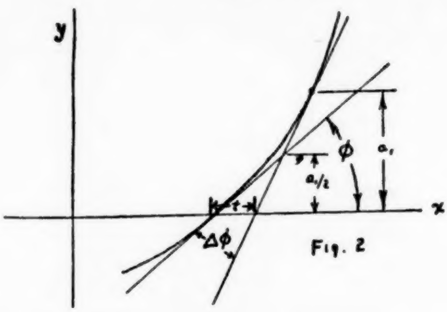

Fig. 2

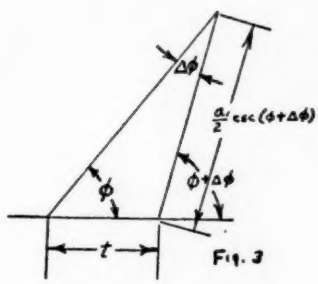

Fig. 3

Assuming constant curvature, for the present, the correction to Newton's formula, t, is given by:

$$t=\frac{a_1}{2}\csc(\phi+\Delta\phi)\sin\Delta\phi\csc\phi$$

$$=\frac{a_1}{2}\Delta\phi\csc^2\phi \qquad \text{approximately}$$

But, $$\Delta\phi=\frac{d\phi}{ds}\Delta s=K\cdot a_1\csc\phi \quad \text{approximately}$$

$$\therefore\ t=\frac{a_1^2}{2}K\csc^3\phi.$$

Now, if K is not constant, but changes by an amount ΔK when x changes from $x=a$ to the approximate value of the root, t changes by a corresponding amount Δt such that,

$$\Delta t=\frac{a_1^2}{2}\Delta K \csc^3\phi.$$

Δt is then the approximate correction to the circle method, or in other words, the approximate error of this method.

Now, $$\Delta K=\frac{dK}{dx}\Delta x$$

and taking $\Delta x=-\dfrac{a_1}{a_2}$, it can easily be shown that,

$$\Delta K=-\frac{(a_2^2+1)6a_4-12a_2a_3^2}{(a_2^2+1)^{5/2}}\cdot\frac{a_1}{a_2}$$

Where $$a_4=\frac{f'''(a)}{3!}$$

$$\tan\phi=\tan(\phi+\Delta\phi)\quad\text{approximately}=a_2.$$

Therefore, $$\csc^3\phi=\frac{(a_2^2+1)^{3/2}}{a_2^3}.$$

Therefore,

(6) $$\text{the approximate error}=\frac{3[2a_2a_3^2-a_4(a_2^2+1)]}{(a_2^2+1)a_2^4}\cdot a_1^3.$$

Since a_2, a_3, and a_4 depend only upon the shape of the curve, and not upon the roots, the coefficient of a_1^3 in (6) may be evaluated with sufficient accuracy from the first approximate value of a. In this manner, the accuracy which can be expected from a given value of a, can be found as soon as $f(a)$ or a_1 is known. Thus, a root may be found to any required degree of accuracy by a single operation with formula (5). It is of course necessary to see if the remaining terms in (3) are negligible.

The above method then consists of the following steps:

(a) Locate the root between two successive integers, and determine a so that $f(a)$ is small, preferably less than unity.

(b) Calculate the coefficient of a_1^3 in (6) and determine a_1 so that the approximate error is sufficiently small.

(c) Redetermine a so that a_1 is small enough.

(d) Obtain the root from (5), checking to see if t_2 in (4) is negligible.

An Application of the Method

An illustration of the method will be furnished by the solution of the equation

$$x^3-2x-5=0$$

to six places of decimals.

(a) Since, $f(2)=-1$, and $f(3)=16$, it is obvious that a root is a little more than 2. Letting $a=2$, we have

(b)
$$\begin{array}{rrrr|l}
1 & +\ 0 & -\ 2 & -\ 5 & 2\ =a \\
 & 2 & 4 & 4 & \\ \hline
1 & 2 & 2 & \multicolumn{1}{|l}{-1=a_1} & \\
 & 2 & 8 & \multicolumn{1}{|l}{} & \\ \hline
1 & 4 & \multicolumn{1}{|l}{10=a_2} & & \\
 & 2 & \multicolumn{1}{|l}{} & & \\ \hline
1=a_4 & 6=a_3 & & &
\end{array}$$

$$\text{The Error term (6)} = \frac{3[2\cdot 10\cdot 36-101]}{1010000}\cdot a_1^3=0.00184a_1^3.$$

For six place accuracy take $\Delta t \leqq 0.0000005$.

Then $a_1 \leqq 0.0656$.

(c) From Newton's formula the root is approximately,

$$2-\frac{-1}{10}=2.1.$$

Taking $a=2.1$, we have by synthetic division:

$$a_1=0.061\ ; \qquad a_2=11.23\ ; \qquad a_3=6.3\ ; \qquad a_4=1\ .$$

Now a_1 is sufficiently small, Δt being equal to 0.0000004 or 0.0000003 if the new values of a_2, a_3, and a_4 are used.

(d) Applying (5),

$$x=2.1-\frac{0.061\cdot 6.3+11.23^2+1}{11.23^2+1}\cdot\frac{0.061}{11.23}$$

$$=2.0945517\cdots\cdots$$

(Second term of series is negligible,

$$=-0.00000002).$$

It can easily be shown that Δt should always be subtracted algebraicly. Correcting for Δt we have:

$$x=2.0945514\cdots\cdots$$

or, accurate to six places of decimals,

$$x=2.094551.$$

A Note on Chords of a Conic

By T. E. RAIFORD
University of Michigan

An elementary problem of analytic geometry in the study of conic sections is the determination of a chord, given its midpoint and the equation of the conic. A method of solution which seems not to have been presented is worthy of note since it makes use of a valuable property of which the student has little chance to become familiar, namely that of reflection.

Let the conic be given by the equation $f(x,y)=Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0$ and (x_0,y_0) be the midpoint of the chord to be found. The common chord of the conics $f(x'+x_0,\ y'+y_0)=0$ and $f(-x'+x_0,\ -y'+y_0)=0$ is the chord desired and its equation referred to the original axes of coordinates is $f(x+x_0,\ y+y_0)-f(-x+x_0,\ -y+y_0)-k=0$.

The proof is immediately obvious since the two conics are each a reflection of the other in the new origin (x_0,y_0), and consequently the points of intersection are symmetric with respect to the origin. The value of k is readily determined since the equation must be satisfied by (x_0,y_0).

The equation of the chord becomes on simplification

$$Ax_0x+B(x_0y+y_0x)+Cy_0y+Dx+Ey-\frac{k}{4}=0.$$

Envelopes Associated with a One-Parameter Family of Straight Lines

By HARRIET F. MONTAGUE
University of Buffalo

In a recent paper by D. H. Richert* it was suggested that after examining the equation $ax+y=0$ to find that it represents a family of straight lines, the student in an elementary course might well examine the equation $a^2+2ax+y=0$. As a varies, the line represented by the equation takes up different positions to form the envelope $y=x^2$. It is the purpose of the present discussion to exhibit the envelope of the general family

$$a^n+na^{n-1}+[n(n-1)/2!]a^{n-2}+[n(n-1)(n-2)/3!]a^{n-3} \\ +\ldots\ldots+nax+y=0$$

in determinant form.

In the case of $a^2+2ax+y=0$, the envelope $y=x^2$ can be written in the form

$$\begin{vmatrix} 1-x & x-y \\ 1 & x \end{vmatrix}=0.$$

Consider next the envelope of

$$f_3(a,x,y)\equiv a^3+3a^2+3ax+y=0.$$

The envelope is found by eliminating a between the above equation and the equation $a^2+2a+x=0$ which is equivalent to $\partial f_3/\partial a=0$. The envelope is

$$\begin{vmatrix} 1 & 3 & 3x & y & 0 \\ 0 & 1 & 3 & 3x & y \\ 1 & 2 & x & 0 & 0 \\ 0 & 1 & 2 & x & 0 \\ 0 & 0 & 1 & 2 & x \end{vmatrix}=0.$$

*D. H. Richert: *Concerning the Teaching of the Linear Equation.* NATIONAL MATHEMATICS MAGAZINE, Vol. XI, No. 8, pp. 382-384.

The expansion of this fifth order determinant can be reduced to the expansion of a third order determinant, and the envelope becomes

$$\begin{vmatrix} 2-2x & x-y & 0 \\ 0 & 2-2x & x-y \\ 1 & 2 & x \end{vmatrix} = 0.$$

This agrees with results already established.*

Consider next the envelope of

$$f_4(a,x,y) \equiv a^4+4a^3+6a^2+4ax+y=0$$

found by eliminating a between this equation and $\partial f_4/\partial a=0$. The envelope in this case is

$$\begin{vmatrix} 3-3x & x-y & 0 & 0 \\ 0 & 3-3x & x-y & 0 \\ 0 & 0 & 3-3x & x-y \\ 1 & 3 & 3 & x \end{vmatrix} = 0.$$

This also agrees with established results.†

The extension can be made to the equation involving the parameter a to the nth degree. The problem is that of finding the eliminant between the equations

$$f_n(a,x,y) \equiv a^n+na^{n-1}+[n(n-1)/2!]a^{n-2}+\ldots\ldots+nax+y=0$$

and

$$(1/n)\partial f_n/\partial a \equiv a^{n-1}+(n-1)a^{n-2}+[(n-1)(n-2)/2!]a^{n-3}$$
$$+\ldots\ldots+(n-1)a+x=0.$$

The equation of the envelope, in the form of an nth order determinant equal to zero, is as follows:

$$\begin{vmatrix} (n-1)-(n-1)x & x-y & 0 & \ldots\ldots\ldots\ldots\ldots\ldots 0 \\ 0 & (n-1)-(n-1)x & x-y & \ldots\ldots 0 \ldots\ldots\ldots 0 \\ 0 & 0 & (n-1)-(n-1)x & x-y \ldots\ldots\ldots 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots\ldots\ldots\ldots (n-1)-(n-1)x & x-y \\ 1 & n-1 & (n-1)(n-2)/2! \ldots\ldots n-1 & x \end{vmatrix} = 0.$$

The order as well as the class of the envelope is equal to n, a coincidence that does not often occur.‡ There is no contradiction here

*George Salmon: *A Treatise on the Higher Plane Curves.* New York. G. E. Stechert & Co., 1934, p. 68.
†Salmon. *loc. cit.*
‡Salmon. *op. cit.* p. 70.

to the established theory, however, since we have taken a very special type of family in our $f_n(a,x,y)=0$, and the results were to be expected in view of the character of that family.

The role of the point (1,1) in both the family and the envelope is an exceptional one. There is only one line of the family through (1,1), that one for which the parameter a is equal to -1. The envelope also passes through the point (1,1); in fact, the point (1,1) is a multiple point of order $n-1$ on the envelope.

Dr. James McGiffert, Rensselaer Polytechnic Institute, has come into possession of a rare and interesting book, through the generosity of Mr. George Owen Knapp of Santa Barbara, California, an alumnus of the class of 1876. The title of the book is *Le Diverse et Artificiose Machine* by Capitano Agostino Ramelli. Ramelli was Engineer to the King of France, who subsidized the publication of the volume. The volume is bound in vellum, the binding being from an Antiphonarium of the 15th Century. The treatise is of the date 1588. Many engineering and maritime devices are described and illustrated by magnificent copper plate engravings. The description of each piece of machinery is given first in Italian, and then in French.

Engineers who have examined this three hundred and fifty year old book are surprised by the ingenuity and modernity of the engineering apparatus used at such an early date.

The book is in a remarkably excellent state of preservation, and the care and skill which obviously combined to produce this work, are a source of wonder to those who have examined it.

It is particularly fitting that this old treatise of Ramelli should be in the hands of Dr. McGiffert, representative of the oldest engineering school in the English speaking world.

A Pre-Euclidean Fragment of the *Elements**

By Allen A. Shaw
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In 1884 the editors of *Pazmaveb*† published a new fragment of geometry with the title: "A newly found Fragment of the Elements of Geometry by Euclid of Alexandria, translated into old Armenian by the ancients." The original MS is at the University of Pavia, Italy, and has neither date nor the name of the translator. It begins with Definition 20 and ends with Proposition III of Book I. By the above title and by the *very brief* account of the life and work of Euclid, the editors *assumed* the fragment to be the work of Euclid. They made no comment on the *text*, but, to clarify a few doubtful and illegible passages in the MS, they published in parallel columns the corresponding text of Euclid in Greek, as shown by the photographs below. No attention has been given to this important fragment since its publication in 1884, and it seems that everyone assumed that it was just *another* fragment of Euclid added to the list of already existing MSS! Strangely enough, Father K. Zarphanalian, in his *History of Ancient Armenian Literature*, p. 586. 4th ed. 1932, Venice, Italy, attributes *this fragment* to Gregory Magistros, an Armenian writer of the 11th century. Victor Langlois‡ also assumes that Gregory Magistros is the translator of Euclid's *Elements*, and does not refer to the Pavia fragment, but to the *Venice* fragment which consists of only one page. Regarding the translation of Euclid's *Elements* by G. Magistros, the editors of the Armenian great *Haigazian* Dictionary (1836), Vol. I, p. 12, are more careful when they say "the translation (of Euclid)

*Read before November meeting of American Mathematical Society at Pasadena, 1938. See *Bulletin* of American Mathematical Society, Vol. 43, p. 774.

†An Armenian journal of international reputation, established in 1843, and is still published by the Mekhitarist Order, Venice, Italy.

‡See *Collection des histoire anciens et modernes de l'Arménie*, p. 403, Vol. I, Paris, 1880.

appears to be the work of GregoryMagistros according to his promise." The last phrase means that Magistros *promised* to translate the *Elements* but there is no evidence that he actually did the work. All he says in his autobiography regarding Euclid is this: "I have *begun* the translation of Euclid's geometry."

Before offering any comment on the Pavia fragment, I propose to give my readers a faithful translation of the MS. and then compare certain passages with the corresponding test in Greek, to derive inevitable conclusions regarding the authorship of the fragment.

TRANSLATION OF THE FRAGMENT

(*Note.—Words in parentheses are not found in the MS.*)

20. Rectilineal figures are those which are contained (*lit.* surrounded by) straight lines.
21. And those that have (three sides) are surrounded by three straight lines.
22. And those that (have) four, are surrounded by four straight lines.
23. And those that (have) more (sides) than these, are likewise surrounded by the necessary number of straight lines.
24. And those that have three sides, there is among them (one) that has three sides equal, that one whose sides are equal.
25 And those which are called isosceles, are those that have only two of its sides equal.
26. And among those having unequal sides, there are those whose three (sides) are unequal to one another.
27. And among those figures, which have three sides, there is one of them (which is) right-angled triangle, that which has the angle right.
28. And obtuse-angled triangle, that which has the angle obtuse.
29. And acute-angled triangle, that which has its three angles acute.
30. And figures, that have four sides, there is among them (the) square, which has its sides equal, and its angles right.
31. There is also (that), which is called oblong, that which has its angles right (Armenian: "oughghovt.")* and its sides unequal.
32. There is again that which is called rhombus (Armenian, *Tevr*, or *Tyur*) that one which has its sides equal, and its angles unequal.

*The translator uses the usual word "oughigh in 22, 23, 27, 30, but here uses another word, both meaning "right."

33. And there is among them the rhomboid (Armenian ***tyur like***) that which has its opposite sides equal and opposite angles equal (*lit*: "two opposite sides"..."two opposite angles"). And there is among them that which is called ***sheghial,**** that which has neither its sides equal, nor its angles right.

34. And among those four sided figures, not mentioned above by us, there is that called trapezium (Arm. ***shegh-like***).

35. And straight lines placed side by side, there are those which, being in the same plane, and if produced to infinity in both directions (*lit.* "sides"), do not meet one another, neither in one nor in the other direction (*lit.* "side").

AND THINGS WHICH WE HAVE TO POSTULATE

(ARM: "to accept as possible," "reasonable")

ARE FIVE IN NUMBER†

1. To draw straight lines from point to point.
2. To produce (*lit*: "to enclose") a straight line...‡
3. To describe a circle with any (*lit*: "every") point (*i. e.* centre) and any (*lit*: "every") m(easure).§
4. That all right angles are equal to one another. (Arm:. "have equality to one another", an idiom of the *fifth* century.)
5. If a straight line falls on two straight lines, and makes the two angles on the same side, which are less than two right angles, when they are produced (the two straight lines) towards the *less* direction⁑ they meet one another.

SCIENCE ACCEPTABLE TO EVERYONE¶

1. Things which are equal to the *same* (*lit*: "one") thing, they themselves are equal to one another.
2 If equals be added to other equals, the wholes become equal.
3. And if equal(s) be subtracted from equal(s), the remainders are equal.
4. And if equal(s) be added to unequal(s), the wholes are unequal.

*This is either continuation of the Def. of Rhomboid or Def. of *another* figure.
†Euclid has a single word for this sentence: 'Αιτηματα.
‡The rest illegible in the fragment.
§"Measure" not legible in MS, only first letter *m* readable.
⁑*i. e.* on the side where the sum of the two angles is less than two right angles!
¶The Greek Test has just *two* words: "common Notions."

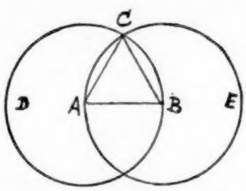
5. And if equal(s) be subtracted from unequal(s), the remainders are unequal.
6. And those (things), each of which is double of the same (thing), are equal.
7. And those, each of which is half of the same (thing), are equal.
8. And those (MS. singular: "that"), when superposed, if they coincide with one another,* are equal.
9. And the whole is greater than its parts.
10. And two straight lines do not enclose a space.†

INTRODUCTION OF THE FIRST BOOK‡

PROPOSITION I

We wish to construct an equilateral triangle on the given straight lines.§

Let *AB* be the given line. And we take *A* as centre, and we describe upon it a circle, with distance (*lit*: "measure") *AB*, that is *BCD*. And we take *B* as centre, and we describe a circle, with distance (*lit*: "measure") *BA*, that is, *ACE*; and we draw *AB* and *BC*, and we say that we have constructed an equilateral triangle on the given line. And its proof is this: *AC* is equal to *AB*, for they are two half-diameters (*i. e.* two radii) of the circles *BCD*, and *BC* is equal to *AB*, for they are two half-diameters of the circle *ACE*. Hence *AC* and *BC* are equal. And this is what we wished to prove.



PROPOSITION II

We wish to place (*lit*: "touch") *at a given point a line like*‡ *the given line.*

*Lit. "And that which is neither more nor less than the other, when superposed, are equal."

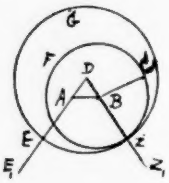
†Note the Armenian MS. has *ten* axioms, while the Heiberg edition has only *five*, and Simson *twelve*; the last two of Simson are the 4th and 5th Postulates of the Pavia MS.

‡*Lit.* "Preface to the First Dissertation."

§In Gk. sing. "finite line."

‡He means *equal* to the given line. There is a definite word for "equal" in ancient Armenian and the translator has used it many times above, but he uses the word *vorbes* = "as," "like," the equivalent of which must have been in *his* Greek Text.

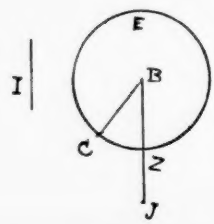
And let A be that point, and BC the line. And let us join A to B, and let us (construct) on it an equilateral triangle, which is DAB; and we produce DA, DB to E and Z. And let us draw with the two centres, B, D, and the two distances ED, BZ, two circles, CZ, ZGE. And the proof is (as follows); DE is like (*i. e.* equal) DZ, and DA is like (*i. e.* equal) DB; amd if equals be subtracted from equals, the remainders are equal.* Hence AE is like BC, which is equal to BC. And this is what we wished ro prove.



PROPOSITION III

We wish to cut off from two given lines, from the greater equal (*lit*: "like") *to the less.*

And let the greater line be BJ, and the less I, and let us construct (*lit*: "join") at the point B, line CB like (*i. e.* equal) line I, and let us describe about centre B, and with distance BC, the circle CZE; and BZ, which it cuts, is like (equal) BC, for they are the two half-diameters of the circle.



If we now compare the above translation (or the original Armenian MS) with the two standard editions of Euclid—Heiberg and Simson—we will note striking and serious variations and differences in the Pavia fragment. We may classify these changes under

(1) *Omissions.* This is the most serious and at the same time the most illuminating change in the MS, *E. g.*, the fragment, instead of giving us the six Euclidean divisions of a proposition (for which see Heath: *Euclid in Greek* pp. 160-1), omits, in Propositions I, II, III, διορισμός, (*definition* or *specification* of a theorem or problem), and

*Neither Simson nor Heiberg quote this in 1.2. Here compare the Gk. Text.

συμπέρασμα, *conclusion*). In Propositions I and II, the Armenian MS omits Axiom 1 ("But things which are equal to the same thing are equal to one another") and in Proposition II, the fragment adds in its place Axiom 5 ("If equals be subtracted from unequals, the remainders are unequal"),—the Aristotelian favorite axiom.

To give the reader some idea of the extent of abbreviations in the Armenian MS, we may compare the Pavia fragment with the Greek text (see below) in the form of the following ratios:

Lines in Arm. MS : lines in Gk. Text = 15 : 20½, (Prop. I)
Lines in Arm. MS : lines in Gk. Text = 13 : 23 nearly, (Prop. II)
Lines in Arm. MS : lines in Gk. Text = 8 : 15 nearly, (Prop. III)

(2) The next variation or change which the reader will note in the Armenian fragment is *lack of precision* in definitions of concepts, consequently *lack of rigor*, *e. g.* the famous Parallel—Postulate is *imperfectly* worded in the Pavia MS;—"If a straight line falls on two straight lines, and makes the two angles on the same side, which are less than two right angles, when they are produced towards the less direction they meet one another", while Euclid has the rigorous and perfect statement: "If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles".

Another example of imperfect statement, in the Armenian text is in the προτασις of Proposition I, where we have "straight lines" (plural), and in the 'έκθεσις, "straight line", while Euclid has the more careful phrase, "*finite* straight line", in both places. Note also that the Arm. MS has not a single word for "parallel", but uses, instead, the phrase,—"And straight lines placed side by side"—another sign of the *primitive* character of the MS.

(3) Still another variation is "half-diameter", twice in Proposition I, and *once* in Proposition III, instead of the usual Euclidean word *distance* as given in Heiberg and Simson editions.

The careful reader will discover other variations for himself when he compares the Armenian text with the Greek.

The variations in (1), (2), and (3) are very important and suggestive. They are *not* improvements on the text of Euclid, on the contrary, they make the text definitely *primitive*, lacking rigor. We know Theon of Alexandria made changes in the text of Euclid (see Heath: *Elements*, Vol. I, Chapter V) but his changes were *not* so radical, and they were intended to *improve* Euclid's text. He certainly would not *mutilate* the first three Propositions of Book I and reduce them to

nearly half of the original size, nor would he make the Parallel-Postulate so imperfect in statement. Nor would he *omit* the word "finite", *twice* in the Greek text, etc. So we may dismiss from our mind Theon with respect to the variations in the Pavia fragment.

In view of the observations made above, the present writer is convinced that (*a*) the Armenian MS is a fragment of pre-euclidean *Elements* written by Leon or Theudius of Magnesia (see Heath: *Manual of Greek Mathematics*, pp. 183-185) whose works on the *Elements* are lost. (*b*) The Armenian MS is the *first* specimen of pre-euclidean *Elements*, preserved through the medium of Armenian translation.

With regard to date of translation of the Armenian fragment close study of style and diction leads the present writer to believe that the translation may be as early as the fifth century.

It must be said that the degree of exactness of the intuition of space may be different in different individuals. perhaps even in different races. It would seem as if a strong naive space-intuition were an attribute pre-eminently of the Teutonic race, while the critical, purely logical sense is more fully developed in the Latin and Hebrew races. A full investigation of this subject, somewhat on the lines suggested by Francis Galton in his researches on heredity, might be interesting.—Felix Klein, *Evanston Colloquium Lectures*, 1893, p. 46.

When a French reader first opens Maxwell's book, a feeling of uneasiness and often even of mistrust mingles at first with his admiration. Only after a prolonged acquaintance and at the cost of many efforts does this feeling disappear. There are even some eminent minds that never lose it.

Why are the English scientist's ideas with such difficulty acclimatized among us? It is, no doubt, because the education received by the majority of enlightened Frenchmen predisposes them to appreciate precision and logic above every other quality.

The old theories of mathematical physics gave us in this respect complete satisfaction. All our masters, from Laplace to Cauchy, have proceeded in the same way. Starting from clearly stated hypotheses, they deduced all their consequences with mathematical rigor, and then compared them with experiment. It seemed their aim to give every branch of physics the same precision as celestial mechanics.

A mind accustomed to admire such models is hard to suit with a theory. Not only will it not tolerate the least appearance of contradiction, but it will demand that the various parts be logically connected with one another, and that the number of distinct hypotheses be reduced to a minimum.—Henri Poincare, *Science and Hypothesis*, p. 148, (George Bruce Halsted translation, 1905).

Thus even at that time that which seems to be denied German society, was unsuccessful, *viz.*, to build up a unified cultural atmosphere which includes the scientific element as a peculiar and matter-of-course component.—Felix Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, p. 100.

κ΄. Εὐθύγραμμα σχήματά ἐστι, τὰ ὑπὸ εὐθειῶν περιεχόμενα.

κα΄. Τρίπλευρα μὲν, τὰ ὑπὸ τριῶν.

κβ΄. Τετράπλευρα δὲ, τὰ ὑπὸ τεσσάρων.

κγ΄. Πολύπλευρα δὲ, τὰ ὑπὸ πλειόνων ἢ τεσσάρων εὐθειῶν περιεχόμενα.

κδ΄. Τῶν δὲ τριπλεύρων σχημάτων, ἰσόπλευρον μὲν τρίγωνόν ἐστι, τὸ τρεῖς ἴσας ἔχον πλευράς.

κε΄. Ἰσοσκελὲς δὲ, τὸ τὰς δύο μόνας ἴσας ἔχον πλευράς.

κς΄. Σκαληνὸν δὲ, τὸ τὰς τρεῖς ἀνίσας ἔχον πλευράς.

κζ΄. Ἔτι τε, τῶν τριπλεύρων σχημάτων, ὀρθογώνιον μὲν τρίγωνόν ἐστι, τὸ ἔχον ὀρθὴν γωνίαν.

κη΄. Ἀμβλυγώνιον δὲ, τὸ ἔχον ἀμβλεῖαν γωνίας.

κθ΄. Ὀξυγώνιον δὲ, τὸ τρεῖς ὀξείας ἔχον γωνίας.

λ΄. Τῶν δὲ τετραπλεύρων σχημάτων τετράγωνον μέν ἐστιν, ὃ ἰσόπλευρόν τέ ἐστι καὶ ὀρθογώνιον.

λα΄. Ἑτερόμηκες δὲ, ὃ ὀρθογώνιον μὲν, οὐκ ἰσόπλευρον δέ.

λβ΄. Ῥόμβος δὲ, ὃ ἰσόπλευρον μὲν, οὐκ ὀρθογώνιον δέ.

λγ΄. Ῥομβοειδὲς δὲ, τὸ τὰς ἀπεναντίον πλευράς τε καὶ γωνίας ἴσας ἀλλήλαις ἔχον, ὃ οὔτε ἰσόπλευρόν ἐστιν, οὔτε ὀρθογώνιον.

λδ΄. Τὰ δὲ παρὰ ταῦτα τετράπλευρα τραπέζια καλείσθω.

λε΄. Παράλληλοί εἰσιν εὐθεῖαι, αἵ τινες ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι, καὶ ἐκβαλλόμεναι ἐπ᾽ ἄπειρον ἐφ᾽ ἑκάτερα τὰ μέρη, ἐπὶ μηδέτερα συμπίπτουσιν ἀλλήλαις.

Ի. (Ձևք ուղղա)գիծ իցեն, որ պատածն գծերեաւն ուղիղ գծոց.

ԻԱ. Եւ (որք երից ուղ)ղին կոյս՝ պատածն յերից ուղիղ գծից.

ԻԲ. Եւ որք չորս՝ 'ի չորից պատածն ուղիղ գծից.

ԻԳ. Եւ որ աւելի քան զչորսն՝ բազմակողմն պատածն բազմութեան ուղիղ գծոցն.

ԻԴ. [illegible] երեքկողմն ունին կոյս, է 'ի նոցանէ [illegible] որ երեք կողքն հաւասարք են, այն՝ որ հաւասար ունին զերեսին [illegible] միմեանց.

ԻԵ. Եւ որ հաւասարասրունք կոչին՝ այն են, որ 'ի նոցին երեսանց՝ երկուս միայն ունին կողս հաւասարս.

ԻԶ. Եւ յանհաւասար կողմացն են որ երեքն անհաւասարք են միմեանց.

ԻԷ. Եւ 'ի սոյն ձևք, որ երիս ունին կողս, կայ 'ի նոցանէ [illegible]անկիւն ուղղանկիւն, այն որ ուղիղ ունի զանկիւնն.

ԻԸ. Եւ լայնանկիւն բութանկիւն, այն որ բութ ունի զանկիւնն.

ԻԹ. Եւ [illegible]անկիւն սրանկիւն, այն՝ որ սուր ունի զերիս անկիւնսն.

Լ. Եւ ձևք՝ որ չորս ունին կողս, կայ 'ի նոցանէ քառակուսի, որ հաւասար ունի զըկողմն, և ուղիղ զանկիւնսն.

ԼԱ. Եւ է՝ որ կոչի երկայնաւոր, այն՝ որ ուղղորդ ունի զանկիւնսն, և անհաւասար զկողս.

ԼԲ. Կայ գործեալ՝ որ կոչի թեւր, այն՝ որ ունի կողս հաւասարս և անկիւնս անուղիղս.

ԼԳ. Եւ 'ի նոցանէ գոյ նման թեւրի, այն՝ որ զերկու կողս 'ի կողմից իւրոց՝ որ ընդդէմ միմեանց են՝ հաւասար ունի, և երկու անկիւնսն ընդդէմ իւրեանց՝ հաւասարք են. Եւ է 'ի նոցանէ՝ որ եղեալ կոչի, այն՝ որ ոչ զկողսն ունի հաւասար, և ոչ զանկիւնսն ուղիղ.

ԼԴ. Եւ որ այլ ես քառակողմ՝ տարաձեւոյ առանցելոցս՝ քառակուսի ձևք՝ կոչին նման չեղոյն.

ԼԵ. Եւ գիծք ուղիղք եզեալք զուգա միմեանց՝ են որ լինին 'ի միում տարածի մակերևոյթի. և եթէ եզանեն յամենայն յերկուց կողմանց՝ չհանդիպին 'ի միմեանս, ոչ յայսկոյս և ոչ յայն.

ΑΙΤΗΜΑΤΑ

α'. Ἠτήσθω, ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.

β'. Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ' εὐθείας ἐκβάλλειν.

γ'. Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι.

δ'. Καὶ πάσας αἱ ὀρθαὶ γωνίαι ἴσαι ἀλλήλαις εἰσί.

ε'. Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλόμεναι αἱ δύο αὗται εὐθεῖαι ἐπ' ἄπειρον συμπεσοῦνται ἀλλήλαις, ἐφ' ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες γωνίαι.

ΚΟΙΝΑΙ ΕΝΝΟΙΑΙ

α'. Τὰ τῷ αὐτῷ ἴσα, καὶ ἀλλήλοις ἐστὶν ἴσα.

β'. Καὶ ἐὰν ἴσοις ἴσα προστεθῇ, τὰ ὅλα ἐστὶν ἴσα.

γ' Καὶ ἐὰν ἀπὸ ἴσων ἴσα ἀφαιρεθῇ, τὰ καταλειπόμενά ἐστιν ἴσα.

δ'. Καὶ ἐὰν ἀνίσοις ἴσα προστεθῇ, τὰ ὅλα ἐστὶν ἄνισα.

ε'. Καὶ ἐὰν ἀπὸ ἀνίσων ἴσα ἀφαιρεθῇ, τὰ λοιπά ἐστιν ἄνισα.

ϛ'. Καὶ τὰ τοῦ αὐτοῦ διπλάσια, ἴσα ἀλλήλοις ἐστί.

ζ'. Καὶ τὰ τοῦ αὐτοῦ ἡμίση, ἴσα ἀλλήλοις ἐστί.

η'. Καὶ τὰ ἐφαρμόζοντα ἐπ' ἄλληλα, ἴσα ἀλλήλοις ἐστί.

θ'. Καὶ τὸ ὅλον τοῦ μέρους μεῖζόν ἐστι.

ι'. Καὶ δύο εὐθεῖαι χωρίον οὐ περιέχουσιν.

ԵՒ ԻՆՉ՝ ՈՐ ԿԱՐՈՑ ԵՄՔ ՀԱՒԱՆԱԿԱՆ ԼԻՆԵԼ. ՎԱՍՆ ՆՈՑԱ՝ ԵՆ ՀԻՆԳ

Ա. Գծեք ուղիղը, զոր հանեմք կետէ 'ի կէտ.

Բ. Ցուցիլ գիծ բովանդ(ակել....)

Գ. Որ յամենայն կէտ և յամենայն չ... (յափ) հանեմք բոլոր.

Դ. Զի ամենայն ուղիղ անկիւնք հաւասարութիւն ունին առ իրեարս.

Ե. Զի եթէ անկանի ուղիղ գիծ 'ի վերայ երկու ուղիղ գծի, և առնէ երկու անկիւն 'ի մի կողմն՝ որ լինին պակաս քան զերկու ուղղորդ, յորժամ ելանեն (երկու ուղիղքն) յայն պակաս կողմն՝ դիպին միմեանց.

ԳԻՏՈՒԹԻՒՆ ՀԱՆՈՒՐՑ ՀԱՒԱՆԱԿԱՆ

Ա. Իրք՝ որ հաւասար են միում իրի, ինքեանք հաւասար են միմեանց.

Բ. Եւ եթէ յաւելու 'ի հաւասարսն՝ այլ եւս հաւասարս, լինին ամենայնքն հաւասարք.

Գ. Եւ եթէ պակասէ 'ի հաւասարէն՝ հաւասար, որ մնայն՝ հաւասարք են.

Դ. Եւ եթէ յաւելու 'ի վերայ անհաւասարին՝ հաւասար, լինին ամենայնքն անհաւասար.

Ե. Եւ թէ պակասէ յանհաւասարէն՝ հաւասար, որ մնայն՝ անհաւասար է.

Զ. Եւ այնք՝ որ իւրաքանչիւր որ կրկին է միոյն՝ հաւասարք են.

Է. Եւ այնք՝ որ իւրաքանչիւրքն կէս են միումն՝ հաւասարք են.

Ը. Եւ այն՝ որ ոչ յաւելու մի քան զմին՝ յորժամ դնի միմեանց 'ի վերայ՝ հաւասարք են.

Թ. Եւ բոլորն՝ մեծ է քան զմասնն.

Ժ. Եւ երկու ուղիղ գիծ՝ ոչ պարագրեն զմակերեւութիւն.

ΠΡΟΤΑΣΙΣ Α΄.

Ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συστήσασθαι.

Ἔστω ἡ δοθεῖσα πεπερασμένη ἡ ΑΒ.

Κέντρῳ μὲν τῷ Α, διαστήματι δὲ τῷ ΑΒ, κύκλος γεγράφθω ὁ ΒΓΔ· καὶ πάλιν, κέντρῳ μὲν τῷ Β, διαστήματι δὲ τῷ ΒΑ, κύκλος γεγράφθω ὁ ΑΓΕ· καὶ ἀπὸ τοῦ Γ σημείου, καθ' ὃ τέμνουσιν ἀλλήλους οἱ κύκλοι, ἐπὶ τὰ Α, Β σημεῖα ἐπεζεύχθωσαν εὐθεῖαι αἱ ΓΑ, ΓΒ. Ἐπεὶ οὖν τὸ Α σημεῖον κέντρον ἐστὶ τοῦ ΓΔΒ κύκλου, ἴση ἐστὶν ἡ ΑΓ τῇ ΑΒ· πάλιν, ἐπεὶ τὸ Β σημεῖον κέντρον ἐστὶ τοῦ ΓΑΕ κύκλου, ἴση ἐστὶν ἡ ΒΓ τῇ ΒΑ· ἐδείχθη δὲ καὶ ἡ ΓΑ τῇ ΑΒ ἴση· ἑκατέρα ἄρα τῶν ΓΑ, ΓΒ τῇ ΑΒ ἐστὶν ἴση· τὰ δὲ τῷ αὐτῷ ἴσα, καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ ΓΑ ἄρα τῇ ΓΒ ἐστὶν ἴση· αἱ τρεῖς ἄρα αἱ ΓΑ, ΑΒ, ΒΓ ἴσαι ἀλλήλαις εἰσίν. Ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον, καὶ συνέσταται ἐπὶ τῆς δοθείσης εὐθείας πεπε- ρασμένης τῆς ΑΒ. Ὅπερ ἔδει ποιῆσαι.

ΠΡΟΤΑΣΙΣ Β΄.

Πρὸς τῷ δοθέντι σημείῳ, τῇ δοθείσῃ εὐθείᾳ ἴσην εὐθεῖαν θέσθαι.

Ἔστω τὸ μὲν δοθὲν σημεῖον τὸ Α, ἡ δὲ δοθεῖσα εὐθεῖα ἡ ΒΓ.

Ἐπεζεύχθω δὲ ἀπὸ τοῦ Α σημείου ἐπὶ τὸ Β σημεῖον εὐθεῖα ἡ ΑΒ· καὶ συνεστάτω ἐπὶ αὐτῆς τρίγωνον ἰσόπλευρον τὸ ΔΑΒ, καὶ ἐκβεβλήσθωσαν ἐπ' εὐθείας τῶν ΔΑ, ΔΒ εὐθεῖαι αἱ ΑΕ, ΒΖ, καὶ κέντρῳ μὲν τῷ Β, διαστήματι δὲ τῷ ΒΓ, κύκλος γεγράφθω ΓΗΘ· Καὶ πάλιν, κέντρῳ μὲν τῷ Δ, διαστήματι δὲ τῷ ΔΗ, κύκλος γεγράφθω ὁ ΗΚΛ.

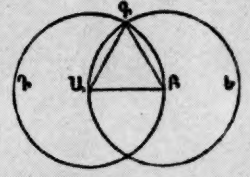
Ἐπεὶ οὖν τὸ Β σημεῖον κέντρον ἐστὶ τοῦ ΓΗΘ κύκλου, ἴση ἐστὶν ἡ ΒΓ τῇ ΒΗ. Καὶ πάλιν, ἐπεὶ τὸ Δ σημεῖον κέντρον ἐστὶ τοῦ ΗΚΛ κύκλου, ἴση ἐστὶν ἡ ΔΛ τῇ ΔΗ, ὧν

ԱՌԱՋԱՐԿՈՒԹԻՒՆ Ա.

Կամիմք ներգործել եռանկիւնս հաւասա_ րակողմս 'ի վերայ տուելոյ գծիցն.

Եւ լիցի տուեալն գիծ ԱԲ.

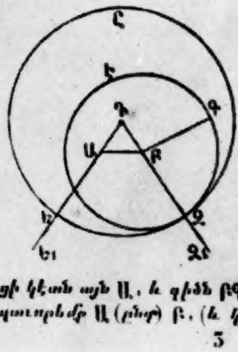
Եւ առնեմք զԱ կենտրոնն, և գծեմք 'ի վերայ նորա բոլոր' 'ի չափ ԱԲին, որ է



ԲԳԴ. և առնեմք զԲ կենտրոնն, և գծեմք բոլոր' 'ի չափ ԲԱին, որ է ԱԳԵ. և հանեմք ԱԳ և ԲԳ, և ասեմք զի արդարեւ եռան- կիւնն հաւասարակող 'ի տուեալ գծէն. Եւ է ցուցումն նորա այս. ԱԳն հաւասար է ԱԲին, վասն զի երկու կէս գիամետրոյն ԲԳԴ բոլորին են. և ԲԳն հաւասար է ընդ ԱԲին, քանզի երկու կէս գիամետրոյն ԱԳԵ բոլորին են. ապա ԱԳն և ԲԳն հաւասար են. Եւ այս է՝ զոր կամէաք ցուցանել.

ԱՌԱՋԱՐԿՈՒԹԻՒՆ Բ.

Կամիմք ձգեցուցանել 'ի տուեալ կէտ՝ գիծ, որպէս տուեալ գիծ.



Եւ լիցի կէտն այն Ա, և գիծն ԲԳ.

Եւ հաստատեմք Ա (ընդ) Բ. (և կազմեմք

ἡ ΔΑ τῇ ΔΒ ἴση ἐστί· λοιπὴ ἄρα ἡ ΑΛ λοιπῇ τῇ ΒΗ ἐστὶν ἴση, ἐδείχθη δὲ καὶ ἡ ΒΓ τῇ ΒΗ ἴση· ἑκατέρα ἄρα τῶν ΑΛ, ΒΓ τῇ ΒΗ ἐστὶν ἴση· τὰ δὲ τῷ αὐτῷ ἴσα, καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ ΑΛ ἄρα τῇ ΒΓ ἐστὶν ἴση.

Πρὸς ἄρα τῷ δοθέντι σημείῳ τῷ Α, τῇ δοθείσῃ εὐθείᾳ τῇ ΒΓ ἴση εὐθεῖα κεῖται ἡ ΑΛ. Ὅπερ ἔδει ποιῆσαι.

ΠΡΟΤΑΣΙΣ Γ.

Δύο δοθεισῶν εὐθειῶν ἀνίσων, ἀπὸ τῆς μείζονος τῇ ἐλάσσονι ἴσην εὐθεῖαν ἀφελεῖν.

Ἔστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι ἄνισοι αἱ ΑΒ, Γ, ὧν μείζων ἔστω ἡ ΑΒ.

Κείσθω πρὸς τῷ Α σημείῳ τῇ Γ εὐθείᾳ ἴση ἡ ΑΔ· καὶ κέντρῳ μὲν τῷ Α, διαστήματι δὲ τῷ ΑΔ, κύκλος γεγράφθω ὁ ΔΕΖ.

Καὶ ἐπεὶ τὸ Α σημεῖον κέντρον ἐστὶ τοῦ ΔΕΖ κύκλου, ἴση ἐστὶν ἡ ΑΕ τῇ ΑΔ· ἀλλὰ καὶ ἡ Γ τῇ ΑΔ ἐστὶν ἴση. ἑκατέρα ἄρα τῶν ΑΕ, Γ τῇ ΑΔ ἐστὶν ἴση· ὥστε καὶ ἡ ΑΕ τῇ Γ ἐστὶν ἴση.

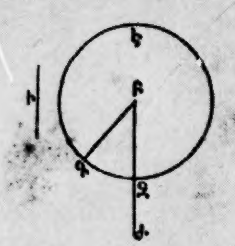
Δύο ἄρα δοθεισῶν εὐθειῶν ἀνίσων τῶν ΑΒ, Γ, ἀπὸ τῆς μείζονος τῆς ΑΒ τῇ ἐλάσσονι τῇ Γ ἴση ἀφῄρηται ἡ ΑΕ. Ὅπερ ἔδει ποιῆσαι.

'ի) վերայ նորա եռանկիւն հաւասարակողմ, որ է ԴԱԲ. և հանեսցի ԴԱ, ԴԲ մինչև յԵ և Զ. և գծեսցի 'ի վերայ երկու կենտրոնին Բ, Դ, և 'ի յերկու հեռաւորութիւնն ԵԴ, ԲԶ, բոլորս երկուս ԳԶԷ, ԶԸԽ. Եւ է զուգումն. ԴԵՆ է որպէս ԴԶՆ, և ԴԱՆ որպէս ԴԲՆ. և եթէ պակասէ 'ի հաւասարէն հաւասար, որ մնայ հաւասար է. ապա ԱԵՆ որպէս ԲԳՆ է, որ հաւասար է ընդ ԲԳՆ. Այս է զոր կամէաք առնել.

ԱՌԱՋԱՐԿՈՒԹԻՒՆ Գ.

Կամիմք հատանել 'ի յերկու անհաւասար գծէն, 'ի յերկայնէն որպէս 'ի կարճէն.

Եւ լինի երկայն գիծն ԲԺ, և կարճն Ի.



Եւ կցեսցի 'ի կէտն Բ՝ գիծ ԲԳ՝ որպէս գիծ Ի. և գծեսցի 'ի վերայ կենտրոնին Բ, և 'ի հեռաւորութիւնն ԲԳ՝ բոլոր ԳՋԷ. և լինի ԲՋ՝ որ հատանէ՝ որպէս ԲԳ. վասն զի են երկու կէս գծամետրութեան 'ի բոլորին.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

The Education of Mathematics Teachers

By WILLIAM L. SCHAAF
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Twenty years ago the standards in vogue for the appointment of mathematics teachers were described* as showing a "very wide divergence;" indeed, the standards in some states were so deplorably low that the National Committee refrained from publishing all the facts. Conditions were so bad that it was deemed impossible to set up any standards that would be generally acceptable. Nevertheless, with an optimistic faith that has since been justified, the Committee, in 1923, with not a little timidity, expressed the hope of approximating a tentative ideal substantially as follows:

(1) *Graduation from a standard four year College or university;*

(2) *Minimum courses in mathematics comprising plane and spherical trigonometry, analytic geometry and the elements of solid analytic geometry, college algebra, calculus, projective geometry, and fundamental concepts of algebra and geometry, totaling in all not less than 33 semester-hours;*

(3) *At least 15 semester-hours of physics and chemistry.*

The mathematical recommendations also expressly called attention to the desirability of including a semester-course each in rational solid geometry, descriptive geometry, analytic projective geometry, theory of statistics, mathematics of investment, surveying, practical and descriptive astronomy, and "in as many other mathematical topics as possible." Subsequent developments (see accompanying table) have shown that, so far as mathematical requirements are concerned, the standards then scarcely hoped for have actually been found possible of attainment, and are essentially comparable to recent recommendations. Thus the joint data of the American Committee of the Inter-

*Reorganization of Mathematics in Secondary Education. Report of the National Committee on Mathematical Requirements. 1923. pp. 506 ff.

national Commission on the Teaching of Mathematics and the National Survey of the Education of Teachers (1934) suggest the clear-cut recommendations in mathematics and related courses as given under Column II of the table; no further comment is necessary. The recommendations (1935) of the special commission appointed by the Mathematical Association of America are summarized in Column III.

TABLE OF REQUIREMENTS FOR MATHEMATICS TEACHERS

	(I) 1923		(II) 1934		(III) 1935	
Pure Mathematics	*Minimum*	*Desirable*	*Minimum*	*Desirable*	*Minimum*	*Desirable*
Freshman Mathematics	6	6	6	6	6	6
Analytic Geometry	3	3	3	6	3	3
Calculus	9	9	6	8	6	6
Projective Geometry	3	3	..	..	..	3
College Geometry	..	3	3	3	3	3
Modern Geometry	..	3	3	6	..	..
Fundamental Concepts	12	12	3	5	3	3
Theory of Equations	..	..	..	..	3	3
Higher Algebra	..	..	..	..	..	3
Advanced Calculus	..	..	..	..	..	6
Total	33	39	24	34	24	36
Related Courses						
Statistics	..	3	3	6	3	3
Economics	..	3	..	..	3	3
Mathematics of Investment	..	3	..	..	3	3
Physics	9	9	..	..	6	9
Other Physical Science	6	12	..	..	6	12
Astronomy	..	3	..	..	..	3
History of Mathematics	..	..	2	3	..	..
Mathematics in Modern Life	..	..	3	3	..	..
Total	15	33	8	12	21	33

(I) Report of National Committee on Mathematical Requirements, 1923.
(II) Joint Data, American Committee of the International Commission, and National Survey of the Education of Teachers, 1934.
(III) Report on the Training of Teachers of Mathematics, by the Commission on the Training and Utilization of Advanced Students of Mathematics, 1935.

Consider now the standards in educational theory and practice teaching. The National Committee's Recommendations (1923) suggested that at least twelve semester-hours be devoted to theoretical professional courses, to be given by teachers of education, and to include history of education, principles of education, methods of teaching mathematics, educational psychology, organization and function of

secondary education. In addition, satisfactory performance of the duties of a teacher of mathematics in a secondary school for a period of not less than twenty semester-hours. About a decade later, conditions began noticeably to improve. Joint* data of the American Committee of the International Commission and the National Survey implied the following standards:

(a) *Introduction to educational concepts, three semester-hours;*

(b) *Psychology, measurements, and other courses in educational theory, nine semester-hours; the teaching of mathematics, (professional treatment of materials), six semester-hours; observation and practice-teaching, three semester-hours.*

Another step forward was taken quickly on the heels of this one. The Commission on the Training and Utilization of Advanced Students of Mathematics, appointed by the Mathematical Association of America, in its Report† on the Training of Teachers of Mathematics, recommended a "one-year course in methods of teaching and practice teaching in secondary mathematics, together with any distinctly pertinent material concerning educational measurements and other content from educational theory (ten semester-hours). It is our belief that this essential part of the student's training should, if possible, be under the direction of professors who have had graduate mathematical training, who have taught mathematics at the secondary level, and who have maintained contacts with the secondary field...... In view of the debatable nature of certain features of our recommendations concerning training in the theory of education and practice teaching, the candidate for a teaching certificate in secondary mathematics is advised of the necessity for satisfying the legal requirements in educational theory as they exist in his locality. We believe that further work in the theory of education beyond this legal minimum, by either an undergraduate or a graduate student, would not be as valuable in preparation for teaching mathematics as additional study of mathematics, related fields and purely cultural subjects." The Report further states: "The utility of these courses in Education probably depends on the qualifications of the instructors and their ideals more largely than is the case in fields where the subject matter is more definitely standardized. However, we believe that in the fields of pure psychology and educational theory there is much material which should be valuable as training for teachers of secondary mathematics. We are inclined to think that, outside of foundation work in psychology, all of the theory

*Sueltz, B. A. *The Status of Teachers of Secondary Mathematics.* 1934. p. 133.
†American Mathematical Monthly, vol. 42, May, 1935, pp. 276-77.

of education presented to the candidate for a secondary teaching certificate in mathematics could best be given in courses definitely oriented with respect to his major teaching field and containing only students whose major or minor interests are in this field."

Conditions did, in short, improve during the ten or twelve years after 1923. According to Sueltz, by 1931 the standards had been raised, in actual practice, to a point where a representative sampling of some 4,000 senior high school mathematics teachers in various parts of the country showed a central tendency of from twenty to twenty-five semester-hours of training in academic mathematics on the college level; about 40% had less than twenty semester-hours, and only about 10% had none at all. As for their professional training, the median number of semester-hours in educational theory was from nineteen to twenty-four; only about 20% of the group had no credit in practice teaching.

Yet as recently as last year, Professor Reeve, in discussing* the desirable qualifications of mathematics teachers, complains that too many of our teachers have a weak academic background, both mathematically and educationally. He goes on to point out that other causes for inefficient and unsatisfactory teaching include a lack of personality, an inexcusable professional apathy, an insularity of mind, and the need for a personal philosophy of education. "If we had a *more cultured* [italics ours] and highly capable group of teachers we should not be hearing so much about the problems relating to the pupil and to content. Obviously, professionally-minded teachers of ability would know very well what to do. Their standards would come from within, whence come, after all, the best standards......" These shortcomings, as we hope to show later, can and must be overcome.

The foregoing account gives a fairly accurate picture of the general trends in the last fifteen years: a reasonably adequate foundation in pure mathematics, stabilized at somewhere around thirty-six semester-hours; an acceptably effective professional preparation, ranging however, from as little as ten to as much as thirty-two semester-hours; and a cultural background not readily expressible in terms of semester-hours, but including at least a modicum of exposure to the physical sciences, to the social sciences, and to literature, languages, and the fine arts.

We come now to the crux of the present discussion. The reader is asked at this point to note the title once more—the *education* of mathematics teachers. In order to secure teachers capable of fulfilling the

*Yearbook of the N. Y. Society for the Experimental Study of Education, 1937, pp. 76 ff.

purposes of secondary education in America today it is not enough to talk in terms of the "training" of such teachers; it is futile to work solely with standards of certification, course requirements, and semester-hours. We must think rather in terms of the *comprehensive education* of the prospective teacher. This viewpoint has been aptly expressed by the Carnegie Foundation in the report of its "Pennsylvania Study," where it is pointed out, in part, that "a candidate must know the specific thing he is to teach, but his eventual worth as a teacher is contained in the perspective of the thing he teaches against the thing he is in his complete intellectual and emotional character. Our present system of pigeonholing a small traditional group of ideas in the mouth of a specialist, then requiring a student to accept and respect those bounds and put together an education out of technical bits contributed by uneducated people, has never worked except in the minds of school and college registrars." Elsewhere in the same report it is suggested that the results of examinations for teaching licenses, however good they may be, cannot possibly be regarded, of themselves, as sufficient to entitle a candidate to a teaching position. "They [examinations] simply elevate him [the candidate] to a platform from which the more difficult choice of those most effective in helping pupils to learn can be made. Educated men and women they must first be. Whether or not young minds will gladly learn from them is another, and, in fact, the all-important matter."

With these opinions the present writer is in complete agreement. Indeed, he is inclined to regard the candidate's entire academic achievement (as expressed in credits, grades, courses, etc.) in pure mathematics, cultural background and educational theory alike, merely as an *initial criterion* in the ultimate selection of the teacher. The eventual solution of the problem of selecting teachers who both know what to teach and are themselves educated, lies with the schools and colleges that prepare them, and not primarily with accrediting agencies, extramural examinations, standards of certification, degrees, or scholastic records. The personnel of the college must assume this responsibility. They know the candidate as no one else can; they have worked with him for four years and are in an enviable position to evaluate his traits, abilities, resources. According to the Carnegie Foundation the college has its in possession adequate information concerning the salient characteristics of the candidate, viz.: (1) what and how much he knows, not merely in courses taken, but the entire content of his mental life; (2) where and how he grows best, and how he uses his knowledge most effectively; and (3) how his total personality behaves under the strain of various situations.

What conclusions, then, may we draw in looking forward to better educated mathematics teachers? In the first place, we approve, in general, of the standards in pure mathematics set up by various agencies as given earlier in this paper; a solid basis of advanced mathematics is undeniably indispensable. However, a note of warning must be sounded right here. Undoubtedly a point is eventually reached beyond which further knowledge of higher mathematics may well cease to contribute to the increased effectiveness of a teacher of high school mathematics, where indeed, it may even prove harmful. Just where that point is may be problematical; perhaps twenty-four semester-hours of mathematics beyond the elementary calculus would be an approximate upper limit, a point above which more training in pure higher mathematics might properly be regarded as of doubtful value. Be that as it may, the question of *quality* is more significant here than that of quantity. In short, it might easily be shown that elective work in the direction of fundamental concepts, theory of equations, history of mathematics, statistics, and theoretical mechanics held greater values than additional study in fields such as the function of a complex variable, group theory, vector analysis, the theory of numbers, analysis situs, or the calculus of variations.

In the second place, we endorse most emphatically any steps taken to insure more adequate and substantial cultural backgrounds, including subject matter in fields related to mathematics. Indeed, we cannot be too insistent upon this point. To be sure, the several mathematical committees whose proposed standards have been set forth above have also called attention to this need. But not urgently enough; this is the nub of the whole matter. Without adequate cultural backgrounds, both mathematical knowledge and professional educational preparation, however complete in themselves, will fail to produce the desired outcome—an educated teacher. In their Report on the Training of Teachers of Mathematics, the special Commission of the American Mathematical Association properly stresses* the desirability of breadth of training rather than specialization, even during the first year of graduate study for those candidates who pursue the master's degree. The commission further recommends "adequate training in English composition and cultural training outside of mathematics and related fields; work in languages, literature, fine arts and the social sciences in preference to increased specialization in mathematics and related fields, and in preference to elective work in the theory of education beyond the legal requirements." We concur heartily, even with the limitation imposed on the extent of educational theory. But the

*American Mathematical Monthly, vol. XLII, May, 1935, pp. 263-277.

recommendation falls down for lack of concreteness; it does not indicate specifically enough how this "cultural training" is to be achieved. This is precisely the point. As has already been suggested, an individual does not necessarily become educated merely by piecing together separate bits of learning.

How, then, is culture to be attained? What practicable means are available to effect the desired integration which produces a cultured person and cultivated personality,—in brief, an educated teacher? It would seem that several possible avenues suggest themselves:

(1) *a more universal and effective use of survey and appreciation courses, not only in mathematics and the physical sciences, but in the social sciences, languages field, and the fine arts as well;*

(2) *greater emphasis upon the history and philosophy of science and of mathematics, with particular attention to interpretation and evaluation rather than merely descriptive history, as suggested by E. T. Bell, Mumford, Sarton, Spengler, Sullivan, Wolf, and others;*

(3) *further development, on the college level, of an "integrated curriculum" or similar arrangement designed to secure, on the part of the student, long range perspective, reflective judgment, a sense of values, critical attitudes, a mature outlook, and receptive sensitivity, together with a profound appreciation of the interrelation and interdependence of the accumulated achievements of human thought;*

a radical reorganization of the work in education theory and method, probably along the lines of "professionalized subject matter," so effected as to be instrumental in revealing and utilizing the individual student's various traits and resources, and to be conducted in terms of the student's individual interests and special aptitudes with a view to the cultivation of desirable attitudes, lively imagination, and fertile resourcefulness;

(5) *as a final means of encompassing the candidate's whole mental life and complete personality, the administrative regimen of the college should be such that the total personal qualifications, scholastic achievement, innate capacities, expressed interests, potential resources and probable future development of the prospective candidate, as manifested throughout his entire academic career, shall be coordinated and appraised by the institution.*

Mathematical World News

Edited by
L. J. ADAMS

Several Benjamin Peirce Instructorships at Harvard University are open for the academic year 1939-40. These instructorships are ordinarily awarded to men who have recently received the Ph.D. degree or have had equivalent training. Those interested in applying should write to the Chairman of the Division of Mathematics.

Professor Arnaud Denjoy of the University of Paris is in residence at Harvard University for the first half of the academic year 1938-39 as Exchange Professor from France.

The annual convention of the Central Association of Science and Mathematics Teachers will be held at the La Salle Hotel in Chicago, on November 25, 26, 1938.

The American Mathematical Society will meet at Western Reserve University and Case School of Applied Science on Friday and Saturday, November 25-26, 1938. By invitation Professor C. C. MacDuffee, of the University of Wisconsin, will deliver an address on ***Modules in algebraic fields***, and Professor V. G. Grove, of Michigan State College, will speak on *A tensor analysis for a V_k in a projective space S_n*.

Professor J. N. Michie, head of the Mathematics Department, informs us that the following appointments to instructorships have been announced at Texas Technological College: Dr. Carrol P. Brady and Miss Lida B. May.

The American Mathematical Society has published two volumes, as a part of the observance of the fiftieth anniversary of the Society. They are:

Volume I. *A Semicentennial History of the American Mathematical Society (1888-1938)*, by Raymond Clare Archibald.

Volume II. *Semicentennial Addresses.*

The American Mathematical Society met at Columbia University, New York City, on Saturday, October 29, 1938. At the afternoon session, Professor G. A. Hedlund of Bryn Mawr College delivered an address on ***The dynamics of geodesic flows.***

The Bulletin of the Calcutta Mathematical Society (India) contains scholarly book reviews, in addition to research papers. A good example is the review of A. A. Albert's *Modern Higher Algebra* by A. C. Choudhury in the March, 1938 issue.

The annual meeting of the British Association was held at Cambridge (England) on August 17-24, 1938. Addresses were divided into several classes, such as topology, theory of groups, geometry and analysis. A highlight of the meeting was the detailed and illustrated account of how tables of functions are prepared, edited and printed.

The thirteenth yearbook of the National Council of Teachers of Mathematics is *The Nature of Proof* by Harold P. Fawcett. The yearbook is a publication issued through the Bureau of Publications, Teachers College, Columbia University (New York City).

The sixth edition of *American Men of Science* (The Science Press) contains the records of 28,000 living men of science.

The Prix Mittag-Leffler was awarded for the first time on June 9, 1937. The recipients were M. Picard (Paris) and M. Hilbert (Göttingen). The award is given in recognition of outstanding contributions to the progress of mathematics. It consists of a gold medal, a diploma and a complete set of *Acta Matematica.*

The Midland Branch of the British Association held its annual meeting on March 11, 1938. Mr. M. A. Porter presented a paper entitled *Bridge Mathematics* in which he discussed the mathematics of contract bridge. Professor G. N. Watson was elected president for the coming session.

Fascicule 89(1938) of the Memorial des Sciences Mathematiques (Gauthier-Villars, Paris) is entitled *Directions de Borel des fonctions méromorphes,* and is the word of M. George Valiron.

Giornale di Matematiche di Battaglini was founded in 1863, and is now edited by Ernesto Pascal (Napoli). It is devoted to research articles, and is published in Napoli, Via Mezzocannone.

Preliminary announcements of the forty-fifth annual meeting of the American Mathematical Society scheduled for Richmond and Williamsburg, Virginia on December 27-30, 1938, list the following addresses:

1. *Intuition, reason and faith in science.* Dean G. D. Birkhoff, Harvard University.

2. *Seismology from the mathematical viewpoint.* Professor W. D. Cairns, Oberlin College.
3. *Fourier expansions of modular functions and theorems on partitions.* Professor H. A. Rademacher, University of Pennsylvania.
4. *On certain abstract spaces.* Professor R. L. Moore.

Dr. L. R. Lieber, director of the Galois Institute of Mathematics, has been appointed visiting professor of mathematics in the graduate school of Duquesne University (Pittsburgh).

Those interested in the applications of mathematics will want to examine *Mathematical Biophysics* by Nicolas Rashevsky, published by the University of Chicago Press. This book is an attempt to lay the foundations for a mathematical biology.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, College Park, Md.

SOLUTIONS

No. 7. Proposed by *T. A. Bickerstaff*, University of Mississippi.

A bullet strikes the face of a wooden beam with a velocity of 1000 feet per second. It is brought to rest at the opposite face after passing through 6 inches of wood in 1/24 second. Find its velocity at the instant when it had passed through 1 inch of wood if the retardation is constant.

Solution by *Robert W. Clack*, Michigan.

Let x = distance traveled by bullet after striking beam, c = constant rate of retardation, and v = velocity when it strikes. Then $d^2x/dt^2=-c$ and $x=vt-ct^2/2$. When $t=1/24$, $x=\frac{1}{2}$ and $dx/dt=0$. That is, $v=c/24$ and $\frac{1}{2}=v/24-c/1152$. Solving gives $c=576$ and $v=24$. But this is inconsistent with the statement that the initial velocity was 1000. Therefore the problem is impossible as stated. With the values of c and v above, it would pass through 1 inch of wood in $(6-\sqrt{30})/24$ seconds and the velocity then would be $4\sqrt{30}$.

No. 20. Proposed by *J. T. Fairchild*, Ada, Ohio.

Integrate the following:

$$S=\int[2a^2+2c^2-2a(a^2+c^2)^{\frac{1}{2}}+(a^2+c^2)\Theta^2]^{\frac{1}{2}}d\Theta.$$

Solution by *C. A. Balof*, Lincoln College, Illinois.

We write $S=b\int[2-2ab^{-1}+\Theta^2]^{\frac{1}{2}}d\Theta$ where $b^2=a^2+c^2$.

Then $$S=\tfrac{1}{2}b\left\{\Theta(\Theta^2+h^2)^{\frac{1}{2}}+h^2\log_e\left[\Theta+(\Theta^2+h^2)^{\frac{1}{2}}\right]\right\}+K,$$

where $h^2=2-2ab^{-1}$ and K is the constant of integration.

No. 42. Proposed by *T. A. Bickerstaff*, University of Mississippi.

Twelve cards have been dealt, six down, and the other six include a jack, two kings, a seven, a five, and a four. What is the probability that the next card will be a four or less?

Solution by *W. R. Volckhausen*, University of Maryland.

There are 46 cards whose positions are unknown. The probability that any one of these 46 cards (as the ace of spades, for example) should occupy the position at the top of the undealt set of forty cards is 1/46. Since 3 of the 46 cards are fours, the probability that the card at the top of the undealt set be a four is 3/46; and that it be a three, two, or ace is, in each case, 4/46.

Thus the solution is 15/46 or 12/46 according as the ace is or is not regarded as less than the four.

The fact that the six cards dealt face down are separate from the forty undealt cards of course does not affect the probability, for they are no more definitely excluded from the next draw than the 39 cards under the top card of the undealt set.

Also solved by *C. A. Balof*.

No. 177. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

It is known that if two circles satisfy the condition:

$$1/(R+d)+1/(R-d)=1/r,$$

where R, r, d are the respective radii and the distance between the centers, then any point on the outer circumference is a vertex of a triangle of which the two circles are respectively the circumcircle and the incircle. Also, that if two circles are repectively the circumcircles and the incircle of a triangle then they satisfy the above equation. Prove that the corresponding equation for a quadrilateral is:

$$1/(R+d)^2+1/(R-d)^2=1/r^2.$$

Solution by *C. A. Balof*, Lincoln College, Illinois.

In Johnson: *Modern Geometry*, Section 125, it is proved that if two circles admit of in – and circumscribed quadrilaterals then the equation above is valid. We outline a proof of the converse.

Let d be the distance beween the centers O and I of two circles of radii R and r respectively. Let the line of centers cut circle O at A and C. Let AM, AN, CP, CQ be tangent to circle I at M, N, P, and Q respectively. Let AM and CQ intersect at D, AN and CP at B, MN and AC at F, and PQ and AC at E. Then triangles ANI and IFN are similar, and AI/IN equals IN/IF, whence IF equals $r^2/(R+d)$. Also, triangles IPC and IPE are similar, and IC/IP equals IP/IE, whence IE equals $r^2/(R-d)$.

Substituting in the given equation, we have $\overline{IE}^2+\overline{IF}^2=r^2$. But from triangle IFN, $\overline{IF}^2+\overline{FN}^2=r^2$. Hence $IE=FN$, and triangles IFN and EIP are congruent. Thus the sum of angles NIF and EIP is $90°$, angle ABC is $90°$ as is also angle CDA. Therefore B and D are on circle O, and any point of circle O may be taken as a vertex of a quadrilateral inscribed in circle O, circumscribed about circle I.

No. 220. Proposed by *G. W. Wishard*, Norwood, Ohio.

The following are easily verified: $33^2=2244$ in the scale of 5; $44^2=3344$ in the scale of 6; $55^2=4444$ in the scale of 7. The law of formation is readily seen; prove the series continued indefinitely.

Solution by *H. T. R. Aude*, Colgate University.

Writing r for the radix and N for the number which is to be squared, we have

$$N=(r-2)r+(r-2)=r^2-r-2.$$

It follows that

$$N^2=r^4-2r^3-3r^2+4r+4=(r-3)r^3+(r-3)r^2+4r+4,$$

which proves the theorem, $r>4$.

Also solved by *H. G. Ayre*, *C. W. Trigg*, and the *Proposer*.

No. 221. Proposed by *E. P. Starke*, Rutgers University.

Prove: if two roots of a cubic equation with rational coefficients have a rational difference, all the roots are rational.

Solution by *W. V. Parker*, Louisiana State University.

By rational transformations the cubic equation may be put in the form

$$x^3+mx+n=0,$$

where m and n are rational. Let the roots be α, $\alpha-r$, $r-2\alpha$, where r is rational. Then from symmetric functions we get

$$3\alpha^2-3r\alpha+r^2+m=0, \tag{1}$$

$$2\alpha^3-3r\alpha^2+r^2\alpha-n=0. \tag{2}$$

If we multiply (1) by $(2\alpha-r)$ and (2) by 3, subtract and solve for α, we find

$$\alpha=\frac{r^3+rm-3n}{2r^2+2m},$$

which is rational. Thus all roots are rational.

The above proof is not valid in case $r^2+m=0$; but in that case also the roots are rational, for it is easy to show that $n=0$ and thus the roots are 0, $\pm r$.

No. 222. Proposed by *Alfred Moessner*, Nurnberg-N, Germany.

What is the general solution in integers of the system:

$$a+b+c+d=e+f+g+h$$

$$a^2+b^2+c^2+d^2=e^2+f^2+g^2+h^2$$

$$a^4+b^4=c^4+d^4?$$

Partial solution by *E. P. Starke*, Rutgers University.

Euler and others have shown how to find numbers satisfying $a^4+b^4=c^4+d^4$; see the account in Dickson, *History of the Theory of Numbers*, Vol. 2, pp. 645-646. We follow Gerardin to take $p=n^7+n^5-2n^3+n$, $q=3n^2$, $r=n^6-2n^4+n^2+1$, $s=3n^5$; $a=p+q$, $d=p-q$, $c=r+s$, $b=r-s$.

Thus for $n=2$, we have $a=158$, $b=59$, $c=133$, $d=134$. From one set p, q, r, s, another set is easily obtained by setting $p'=p+q+r+s$, $q'=p+q-r-s$, $r'=p-q+r-s$, $s'=p-q-r+s$.

Numbers to satisfy the other proposed equations are easily obtained by putting

$$a+b=c+d-2k,$$

$$e=c-k,\ f=d-k,\ g=a+k,\ h=b+k.$$

Thus for the numerical example cited above, we have $k=25$; $e=108$, $f=109$, $g=183$, $h=84$.

Complete generality cannot be claimed for the above method.

No. 223. Proposed by *Albert Farnell,* Louisiana State University.

Form numbers the sum of whose digits is 27 and which are perfect cubes in every number system in which the base is sufficiently large. Find the system of smallest base in which such a cube can be written.

Solution by *G. W. Wishard,* Norwood, Ohio.

These cubes must be computed without any "carrying". Hence, since $3^3=27$, the sum of the digits in the root must be 3. We therefore try the following, r being the base:

$$(r^n+2)^3=r^{3n}+6r^{2n}+12r^n+8,$$

$$(2r^n+1)^3=8r^{3n}+12r^{2n}+6r^n+1,$$

$$(r^n+r^m+1)^3=r^{3n}+3r^{2n+m}+3r^{n+2m}+6r^{n+m}+3r^{2n}+3r^{2m}+3r^n+3r^m$$
$$+r^{3m}+1,$$

$$(3r^n)^3=27r^{3n}.$$

If in any expansion no two exponents agree, the maximum digits of 12, 12, 27, 6 require bases of at least 13, 13, 28, 7. Thus 7 is the minimum base. Thus $1011^3=1{,}033{,}364{,}331$ and $11001^3=1{,}331{,}363{,}\text{-}033{,}001$. (But if, for instance, $n=2$ and $m=1$, the exponents $n+m$ and $3m$ are the same, making a digit 7; e. g., $111^3=1{,}367{,}631$.)

No. 227 Proposed by *Albert Farnell,* Louisiana State University.

A perfectly elastic ball is projected from a certain height above the horizontal plane at an angle of 0° and with an initial velocity of 60 feet per second. If it strikes the plane at an angle of 60° and rises to two-thirds the preceding height with each bounce, find the total area covered in the plane in which it moves, disregarding air resistance.

Solution by *Michael Wales,* Worchester Polytechnic Institute.

We have $x=60t$, $y=h-gt^2/2$. Since, for $y=0$, $dy/dx=\sqrt{3}$ (disregarding signs), we must have $dy/dx=-gx3600=\sqrt{3}$ and, taking $g=32$ ft/sec², $h=169$ feet. Considerations of symmetry demand that $|dy/dx|=\sqrt{3}$, for each of the parabolas described by the ball, when $y=0$.

The area under the parabola $y=k-bx^2$ from $-a$ to $+a$ is easily found to be: $4ak/3$. The parabolas of the problem have for their equation, $y=k-3x^2/4k$, where k represents the height to which the ball bounces. Their areas are $8k^2/3\sqrt{3}$. The total area covered by the ball is then:

$$A=4h^2/3\sqrt{3}+8h^2/3\sqrt{3}\,[4/9+16/81+64/729+\cdots],$$

h being the original height 169 feet. The series in the bracket is geometric with common ratio 4/9. Summing, we find

$$A = 57200 \text{ sq. ft., approximately.}$$

No. 229. Proposed by *Walter B. Clarke*, San Jose, California.

Locate the vertices of a square inscribed to a given triangle so that one side of the square shall be coincident with one side of the triangle, produced if necessary.

Solution by *C. W. Trigg*, Los Angeles City College.

Method I. Construct BE parallel to AC and equal to the altitude BD. Draw AE intersecting BC at F. Draw FG parallel to AC and meeting AB at G. Drop GH and FK perpendicular to AC. $FGHK$ is the required square.

Proof: Draw ED and FH. In ΔABE, $AG:GB::AF:FE$. In ΔABD, $AG:GB::AH:HD$. Hence, $AF:FE::AH:HD$. Therefore, HF is parallel to DE. Now since their sides are respectively parallel, $\Delta BDE \sim \Delta GHF$. But ΔBDE is an isosceles right triangle so ΔGHF is an isosceles right triangle and $GF = GH$. Now the opposite sides of $FGHK$ are parallel so $FGHK$ is a square.*

Method II. Construct the square $ACMN$ on the side AC. Draw BN and BM intersecting AC at H and K, respectively. Erect perpendiculars to AC at H and K meeting AB and BC at G and F respectively. Draw GF forming the required square $FGHK$.

Proof: $GH:AN::BH:BN::BK:BM::FK:CM$. Since $AN = CM$, $GH = FK$. $GH:AN::BH:BN::HK:NM$. Since $AN = NM$, $GH = HK = FK$. It follows from the construction that $FGHK$ is a square.

Discussion. Let x be the side of the "inscribed" square. Then from Figure I it may be seen that $AG/x = c/h_b$ and $GB/x = c/b$. Since $AG + GB = c$, when these equations are added, $1/x = 1/b + 1/h_b$. That is, the reciprocal of the side of the inscribed square is equal to the sum of the reciprocals of the side on which two vertices lie and of the altitude to that side. Hence if the base and altitude of a variable triangle remain constant, the inscribed square with two vertices on that side remains constant. Otherwise, as the square slides along AC, B describes a line parallel to AC.

*This figure, together with suggestions for solution of the problem, appears in Durell, "Plane and Solid Geometry," (1911), p. 228, Exercise 19.

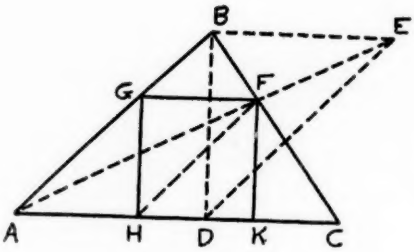

Figure I

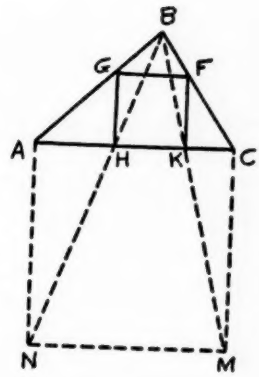

Figure II

Since three such squares may be "inscribed" to a triangle, upon adding the three reciprocal equations, $1/x_a+1/x_b+1/x_c=1/a+1/b+1/c+1/h_a+1/h_b+1/h_c=1/a+1/b+1/c+1/r$. That is, the sum of the reciprocals of the sides of the inscribed squares is equal to the sum of the reciprocals of the sides and of the inradius.*

When the original reciprocal equations are cleared of fractions, $x_b(b+h_b)=bh_b=2\Delta=x_a(a+h_a)=x_c(c+h_c)$. So the product of the side of the square by the sum of the base and altitude of the triangle is equal to twice the area of the triangle. Now $h_b=2\Delta/b$, so $x_b=2\Delta b/(2\Delta+b^2)$, which expresses the side of the square in terms of the sides of the triangle if Heron's formula for Δ be taken.

If the two relations first given be divided, $GB : AG :: h_b : b$, so that the side of the square parallel to the base divides the lines through the vertex in the ratio of the altitude to the base. Furthermore, the area of the triangle cut off by this line is to the area of the given triangle as the inscribed square is to the escribed square. (Figure II). This follows since $x/b=GB/c=BF/a$, whence

$$\frac{x^2}{b^2}=\frac{GB\cdot BF}{ac}=\frac{\Delta GBF}{\Delta ABC}$$

*The sum of the reciprocals of the altitudes of a triangle is equal to the reciprocal of the inradius. Altshiller-Court, "College Geometry," page 72.

PROPOSALS

No. 38. Proposed by *E. M. Shirley*, Louisiana Polytechnic Institute.

An arc light a units high stands in the middle of a street while a man b units high $(b<a)$ walks along a sidewalk at c feet per second. State the parametric equations which give the position of the end of the man's shadow at any time t. Along what locus does the end of the shadow move and at what velocity along the locus? Show analytically that the man's velocity must always be less than that of the end of the shadow.

No. 57. Proposed by *R. B. Thompson*, Beaver Crossing, Nebraska.

Determine a point such that the sum of the nth powers of the distances to three fixed points is a minimum.

No. 251. Proposed by *Yudell Luke*, University of Illinois.

A sphere of radius r has its center on the surface of another sphere of radius R with $2R \geq r$. If the area common to both spheres equals one-half the area of the sphere of radius r, find in terms of r. If R is constant, when is the common area a maximum?

No. 252. Proposed by *Walter B. Clarke*, San Jose, California.

P is any point not on the line XY. $A, B, C, D \ldots$ are points on XY which for convenience are all taken on the same side of the foot of the perpendicular from P to XY. Using an arbitrary angle θ for base angle, similar isosceles triangles are formed on PA, PB, PC, etc., (they may be on either side but all on same side), giving apexes A', B', C',.... Show (1) A', B', C',... are collinear; (2) this line makes an angle θ with XY.

No. 253. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

Point out an analogous feature between the center of mass of the perimeter of a triangle and the center of mass of the surface of a tetrahedron.

No. 254. Proposed by *M. S. Robertson*, Rutgers University.

What is the value of

$$\operatorname*{Lim\ superior}_{n\to\infty}\ \operatorname*{Max}_{0\leq\theta\leq\pi/3}\left|\frac{\cos n\theta}{\cos\theta}\right| ?$$

No. 255. Proposed by *V. Thébault*, Le Mans, France.

An arbitrary point M is taken within a square P. The projections of M upon the sides of the square are the vertices of a quadrangle P_1, called the first pedal of M. The projections of M upon the sides of P_1 form the quadrangle P_2, the second pedal of M.

(1) Show that every fourth pedal, P_{4n}, is again a square.

(2) Calculate the sum of the areas of P, P_4, $P_8, \ldots$.

(3) What are the angles made by the homologous sides of P and P_4, P_4 and P_8, etc.?

No. 256. Proposed by *C. E. Springer*, University of Oklahoma.

Given a plane curve with the equation $y=y(x)$, regular in the neighborhood of the point $P(x_0,y_0)$, and a chord with end-points $A(x_0+dx,y_0+dy)$ and $B(x_0+\delta x,y_0+\delta y)$, parallel to the tangent to the curve at P. If the distance between the chord and tangent is ϵ, prove that

$$\lim_{\epsilon\to 0}\frac{dx+\delta x}{\epsilon}=\pm(2/3)\frac{(d^3y/dx^3)\sqrt{1+(dy/dx)^2}}{(d^2y/dx^2)^2}.$$

No. 257. Proposed by *W. V. Parker*, Louisiana State University.

(1) If $A_1(x_1,y_1)$, $A_2(x_2,y_2)$, $A_3(x_3,y_3)$ are the vertices of a triangle and a_i is the length of the side opposite A_i, show that the equation of the circumscribing circle is:

$$a_1^2\begin{vmatrix} x & y & 1\\ x_1 & y_1 & 1\\ x_2 & y_2 & 1\end{vmatrix}\cdot\begin{vmatrix} x & y & 1\\ x_3 & y_3 & 1\\ x_1 & y_1 & 1\end{vmatrix}$$
$$+a_2^2\begin{vmatrix} x & y & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}\cdot\begin{vmatrix} x & y & 1\\ x_1 & y_1 & 1\\ x_2 & y_2 & 1\end{vmatrix}$$
$$+a_3^2\begin{vmatrix} x & y & 1\\ x_3 & y_3 & 1\\ x_1 & y_1 & 1\end{vmatrix}\cdot\begin{vmatrix} x & y & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=0.$$

(2) Prove that there is one, and only one, ellipse circumscribing the triangle $A_1A_2A_3$ for which this triangle is an inscribed triangle of maximum area. Prove that the equation of this ellipse is:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ x_2 & x_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

(3) Show that an hyperbola having its center at the centroid of a triangle cannot pass through all three vertices of the triangle.*

No. 258. Proposed by *E. P. Starke*, Rutgers University.

When n is a positive integer, the coefficients in the binomial theorem may be divided into five groups as follows:

(1) ${}_nC_0, {}_nC_5, {}_nC_{10}, {}_nC_{15}, \ldots$; (2) ${}_nC_1, {}_nC_6, {}_nC_{11}, {}_nC_{16}, \ldots$;

(3) ${}_nC_2, {}_nC_7, {}_nC_{12}, {}_nC_{17}, \ldots$; (4) ${}_nC_3, {}_nC_8, {}_nC_{13}, {}_nC_{18}, \ldots$;

(5) ${}_nC_4, {}_nC_9, {}_nC_{14}, {}_nC_{19}, \ldots$.

Prove: if the terms in each set are added, the five sums will have just three distinct values; the three possible differences of distinct sums are three consecutive terms of the Fibonacci series, 1,1,2,3,5,8,13,..., in which $a_{n+1} = a_n + a_{n-1}$.

*See prob. 3866, Am. Math. Monthly, March, 1938, p. 190.

Bibliography and Reviews

Edited by
P. K. Smith and H. A. Simmons

Review Course in Algebra. By W. E. Sewell. D. C. Heath, New York, 1938. vi+145 pages.

The author of this book states that it is designed for use in the latter part of the high school curriculum or for independent review work. The book would also be useful for a half-year course in elementary algebra in those colleges which start their mathematics on this elementary level.

The topics included in this book are ones which may be found in almost every elementary algebra. The principal topics treated beyond *quadratic equations* are *systems involving quadratics, progressions, mathematical induction,* and *the binomial theorem.*

This book is to be commended in general for its accuracy of statement, its ample supply of well graded problems, and its clarity of exposition. In particular, the chapter on exponents and radicals finds favor with this reviewer.

The book may be generally criticized on the score of lack of emphasis and too few problems on the topic of *factoring,* and brevity of exposition on certain important topics. It is this reviewer's opinion, for example, that few students in want of review in algebra could master *the theory of mathematical induction* from the scant treatment in this book without the aid of a competent teacher or an adequate text. Other objections may be a matter of personal preference. For example, the reviewer dislikes the use of the word *cancellation* in connection with *simplification of fractions,* and would like to see *graphs* incorporated with the *treatment of linear equations* and the material immediately following.

Northwestern University. H. L. GARABEDIAN.

The Importance of Certain Concepts and Laws of Logic for the Study and Teaching of Geometry. By Nathan Lazar. George Banta Co., Menasha, Wisconsin, 1938. 65 pages. $1.00.

On the assumption that the primary purpose for teaching geometry is to train the mind in logical thinking, the author attempts to

show that "it is possible to utilize geometry as a medium for making the pupils conscious of the existence of logical patterns of valid and and invalid reasoning in mathematics as well as in the thinking of everyday life."

In Chapter I he reviews the history of the concept, "converse of a theorem." He lists the various definitions that have been used and points out the objections to each. He then proposes the new definition: "The converse of a theorem may be obtained by interchanging any number of conclusions with an equal number of hypotheses." It is urged that converse theorems be obtained mechanically from a complete list of the hypotheses and the conclusions of the given theorem rather than from the verbal statement.

Similar studies of the concepts "inverse" and "contrapositive" are given in Chapters II and III, respectively. In Chapter IV a new Law of Contraposition is stated as follows: "If a theorem contains more than one hypothesis and only one conclusion, it is equivalent to any theorem formed from it by the interchange of the contradictory of one of the hypotheses and the contradictory of the conclusion." The author shows how this law may be used to eliminate many indirect proofs, to facilitate independent research, and to simplify the proof of locus problems. In Chapter V the law of Converses is developed and generalized (the author believes for the first time), and applied to geometric problems. It is urged that the important mathematical concepts of *necessary and sufficient conditions* be introduced into the geometry course and that the Law of Contraposition be used to clarify the student's ideas of the meanings of these terms.

The book is well written, relatively free of errors, and is easy to read. Many illustrations of the application of the definitions and laws to geometrical theorems are given and there is a summary at the end of each chapter. The book also contains quite a few *pedagogical hints* and a good bibliography. Although it seems unlikely that the author has found a cure-all for the ills that beset the teacher of elementary geometry, there is much food for thought and many worth-while suggestions in this small volume. It is well worth the reading to every teacher of geometry.

Louisiana Polytechnic Institute. F. C. GENTRY.

College Algebra. By Edwin R. Smith. The Cordon Company, Inc., New York, 1938. 307 pages+49 pages of tables.

This text provides a lucid review of the fundamentals of high school algebra, followed by adequate treatments of most of the con-

ventional topics in *college algebra.* A brief introduction to the elementary concepts of statistics is one of its newer features, and is to be commended. No formal consideration is given to mathematical induction, although the binomial theorem is proved (except for a detail) by means of it in §74.

The review material is followed by a development of the function concept. In this connection, excellent use is made of graphical materials.

Graphical methods are used to good advantage throughout the book. In the theory of linear and quadratic equations, the reviewer feels that the important rôle of graphs is to clarify the theory, rather than to produce approximate roots.

The theory of determinants is not considered in the earlier part of the book; yet it is given in advance of the chapter on permutations. This arrangement causes the explanation of inconsistent and dependent systems of linear equations to rest largely on graphs, since the algebraic formulas which would naturally be represented by determinants are not easily remembered as polynomials. The theory of determinants is based on the principles of permutations, with which a student will have had no previous opportunity to become familiar. The explanation of the method of expanding determinants by minors is especially brief. It is doubtful that the student will find the explanation satisfactory and complete.

The theory of equations of higher degree than the second is treated in two chapters, the first of which affords a clear and unimpeded approach to the problem of obtaining the rational roots of a given equation. A method is also given therein for approximating the irrational roots of an equation. This method could have been improved greatly by devoting a few lines to the straight line method of interpolation.

Chapter XXI is, for an algebra text, an unusually extensive treatment of infinite series. While the theory of infinite series is important, one does not often find time enough to cover such a chapter, except at the sacrifice of topics which seem to be more naturally a part of algebra.

The problems listed appear to be well graded, and the answers given for those in several random samples were found to have a good percentage of accuracy. Unfortunately, the answers to Excercises 3 and 5 on page 6 are incorrect.

One is discouraged to find appearing again the often repeated problems on mixing alcohol and water by measures of volume. Will the mathematicians never learn that a quart of alcohol and a quart of water do not form two quarts of mixture? Student chemists know this, and wink at the mathematical seriousness of authors and teachers who do not.

Definitions in the text are not all that might be expected. Perhaps the worst samples noted are the following: "Equations which have polynomials in both members are called *rational integral equations.* One of the members may be a constant, including zero. It is essential, however, that there be no variable fractions or radicals in either member" (§36, p. 57), and "A sequence of numbers is a set of numbers arranged to some law" (§78, p. 136). Like Plato's definition of a man, to wit, a featherless biped, such definitions as the above might include a plucked rooster. The first definition does not exclude logarithms, exponentials, etc.; while the second one is as descriptive of a magic square, or a matrix, as it is of a sequence.

The editors and proof readers have left a good many small errors. A few of the split infinitives left do not justify themselves on the grounds of added clearness, or smoothness. (Cf., for one example, the split infinitive on page 60, line 12 from the bottom.)

In résumé, the text has more good points than bad ones. A discerning teacher should be able to produce good results from it by doing a judicious amount of editing in the class room.

Central Y. M. C. A. College, Chicago. G. D. GORE.

Intermediate Algebra. By W. E. Brooke and H. B. Wilcox. Farrar and Rinehart, New York, 1938. viii+323 pages. $1.90.

In the preface the authors state: "The object of the book is to bridge the ever-widening gap which exists between a beginning course and college algebra. ... The writers have therefore endeavored to present algebra as a generalization of arithmetic and to lead the student by easy stages from that which is within his experience to that which is not." It is this reviewer's opinion that the authors have accomplished these objectives in a very commendable manner. There are two features of the book which will appeal to many teachers. Each chapter is a lesson, and many of the common errors made by beginning students are exhibited by the use of boxed displays. A sampling check of the answers, which are given after most of the problems, revealed no mistakes. A spot examination of the Index discloses one error, viz., the page reference to the binomial theorem is confused with its section number.

Northwestern University. J. F. KENNEY.

Calculus. By Edward S. Smith, Mayer Salkover, Howard K. Justice. John Wiley and Sons, Inc., New York, 1938. xii+558 pages.

According to the preface, this book has been planned as a first course in calculus "sufficiently practical, and at the same time adequately rigorous, for both technical and non-technical schools." It differs markedly from many such books in arrangement. The fundamental ideas of the differential calculus as applied mainly to polynomials are presented in Chapter II, pages 12-50, and the corresponding ideas of the integral calculus are developed in Chapter III, pages 51-93.

In Chapter II discussion is given of the meaning of a derivative and its applications in finding slopes, speed, acceleration, critical points and points of inflection; and differentials are applied in making approximations, including the computation of relative error and percentage of error. In Chapter III one finds definitions of indefinite and definite integrals, developments of the *fundamental theorem* and *Duhamel's theorem*, and applications of definite integrals in the computation of areas, volumes, pressure, work, and in the determination of geometrical centers. Chapters II and III are expected to give the student a realization of what the calculus will do rather than an impression that the subject is primarily a body of formulas to be manipulated more or less skillfully.

After Chapter III come the derivatives of the various standard forms for differentiation and further applications to problems in geometry, mechanics, and physics. More than one hundred pages are given to integration of forms other than polynomials and to the solution by means of integration of problems in physics and mechanics. Quite a full treatment of series, hyperbolic functions, partial differentiation, and multiple integrals is included. There is no work on differential equations. A short table of integrals and four pages of formulas from algebra, geometry, trigonometry, and analytic geometry are included.

The book calls for more than the usual amount of repetition, which, according to present day psychology, should improve the quality of the student's work; but there is considerable doubt as to whether it would be possible to cover in a one-year course all the material that is given. There is a large number of well chosen problems.

Iowa State College. MARIAN E. DANIELLS.

[*Editorial Note.* Answers are given both to the even and to the odd numbered problems. In a brief examination of these answers, we solved the following problems selected at random and found their answers to be correct: Numbers 12, p. 17; 12, p. 28; 3, p. 38; 3, p. 53, and 3, p. 59. H. A. SIMMONS.]